

## ON CROSS PRODUCT HOPF ALGEBRAS

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ABSTRACT. Let  $A$  and  $B$  be algebras and coalgebras in a braided monoidal category  $\mathcal{C}$ , and suppose that we have a cross product algebra and a cross coproduct coalgebra structure on  $A \otimes B$ . We present necessary and sufficient conditions for  $A \otimes B$  to be a bialgebra, and sufficient conditions for  $A \otimes B$  to be a Hopf algebra. We discuss when such a cross product Hopf algebra is a double cross (co)product, a biproduct, or, more generally, a smash (co)product Hopf algebra. In each of these cases, we provide an explicit description of the associated Hopf algebra projection.

## INTRODUCTION

Given algebras  $A$  and  $B$  in a monoidal category, and a local braiding between them, this is a morphism  $\psi : B \otimes A \rightarrow A \otimes B$  satisfying four properties, we can construct a new algebra  $A \#_{\psi} B$  with underlying object  $A \otimes B$ , called cross product algebra. If  $\mathcal{C}$  is braided, then the tensor product algebra and the smash product algebra are special cases. A dual construction is possible: given two coalgebras  $A$  and  $B$ , and a morphism  $\phi : A \otimes B \rightarrow B \otimes A$  satisfying appropriate conditions, we can form the cross product coalgebra  $A \#^{\phi} B$ .

Cross product bialgebras were introduced independently in [6] (in the category of vector spaces) and in [3] (in a general braided monoidal category). The construction generalizes biproduct bialgebras [12] and double cross (co)product bialgebras [8, 10]. It can be summarized easily: given algebras and coalgebras  $A$  and  $B$ , and local braidings  $\psi$  and  $\phi$ , we can consider  $A \#_{\psi}^{\phi} B$ , with underlying algebra  $A \#_{\psi} B$  and underlying coalgebra  $A \#^{\phi} B$ . If this is a bialgebra, then we call  $A \#_{\psi}^{\phi} B$  a cross product bialgebra. Cross product bialgebras can be characterized using injections and projections, see [3, Prop. 2.2], [6, Theorem 4.3] or Proposition 7.1.

If  $A \#_{\psi} B$  is a cross product algebra, and  $A$  and  $B$  are augmented, then  $A$  is a left  $B$ -module, and  $B$  is a right  $A$ -module. Similarly, if  $A \#^{\phi} B$  is a cross product coalgebra, and  $A$  and  $B$  are coaugmented, then  $A$  is a left  $B$ -comodule, and  $B$  is a right  $A$ -comodule, we will recall these constructions in Lemmas 2.2 and 2.4. In [3], an attempt was made to characterize cross product bialgebras in terms of these actions and coactions. A Hopf datum consists of a pair of algebras and coalgebras  $A$  and  $B$  that act and coact on each other as above, satisfying a list of compatibility conditions, that we will refer to as the *Bespalov-Drabant* list [3, Def. 2.5]. If  $A \#_{\psi}^{\phi} B$  is a cross product bialgebra, then  $A$  and  $B$  together with the actions and coactions from Lemmas 2.2 and 2.4 form a Hopf pair, [3, Prop. 2.7]. Conversely, if we have

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a Hopf pair, then we can find  $\psi$  and  $\phi$  such that  $A \#_{\psi}^{\phi} B$  is cross product algebra and coalgebra, but we are not able to show that it is a bialgebra, see [3, Prop. 2.6]. Roughly stated, the Bespalov-Drabant list is a list of necessary conditions but we do not know whether it is also sufficient.

The main motivation of this paper was to fill in this gap: in Sections 4 and 5, we will present some alternatives to the Bespalov-Drabant list, consisting of necessary and sufficient conditions. Our first main result is Theorem 4.6, in which we provide a set of lists of necessary and sufficient conditions, in terms of the local braidings  $\phi$  and  $\psi$ . Another set, now in terms of the actions and coactions, will be given in Theorem 5.4.

As we have already mentioned, smash product algebras are special cases of cross product algebras, and they can be characterized, see Section 3. In Section 6, we first show that a cross product bialgebra is a smash cross product bialgebra if and only if  $\psi$  satisfies a (left) normality condition, see Definition 6.1. In this situation, the necessary and sufficient conditions from Theorems 4.6 and 5.4 take a more elegant form, see Theorem 6.4. We have a dual version, characterizing smash cross coproduct bialgebras (with cross product coalgebra as underlying coalgebra), and a combination of the two versions yields a characterization of Radford's biproducts, see Corollary 6.3: a cross product bialgebra is a Radford biproduct if  $\psi$  is conormal and  $\phi$  is normal. In Theorem 6.4, we also present sufficient conditions for a smash cross product bialgebra to be a Hopf algebra. All this results have a left and right version; combining the left and right version, we have the following interesting application, see Corollary 6.7: a cross product bialgebra is a double cross product in the sense of Majid if and only if  $\phi$  is left and right normal. In this situation,  $\phi$  coincides with the braiding of  $A$  and  $B$ . Otherwise stated: Majid's double cross product bialgebras are precisely the cross product bialgebras for which the underlying coalgebra is the cotensor coalgebra. Consequently, in the category of sets any cross product Hopf algebra is a bicrossed product of groups in the sense of [15], see Corollary 6.8.

We have already mentioned that cross product bialgebras can be characterized using injections and projections. The aim of Section 7 is to study this characterization in the case of smash cross product algebras. The structure of Hopf algebras with a projection was described completely by Radford in [12]: if  $H$  and  $B$  are Hopf algebras, and there exist Hopf algebra maps  $i : B \rightarrow H$  and  $\pi : H \rightarrow B$  such that  $\pi i = \text{Id}_B$ , then  $H$  is isomorphic to a biproduct Hopf algebra. Several generalizations of this result have appeared in the literature. In [13], the condition on  $\pi$  is relaxed: if  $\pi$  is a left  $B$ -linear coalgebra map then  $H$  is isomorphic to a smash product coalgebra, with an algebra structure given by a complicated formula that does not imply in general that  $H$  is isomorphic to a crossed product bialgebra. The situation where  $\pi$  is a right  $B$ -linear coalgebra morphism was studied with different methods in [2]. The situation where  $\pi$  is a Hopf algebra morphism and  $i$  is a coalgebra morphism is studied in [4], and the case where  $\pi$  is a morphism of bialgebras and  $i$  is a  $B$ -bilinear algebra map is studied in [1].

With these examples in mind, we have been looking for the appropriate projection context on a Hopf algebra, that ensures that the Hopf algebra is isomorphic to a smash cross product Hopf algebra. Here the idea is the following. If  $H = A \#_{\psi}^{\phi} B$  is a cross product coalgebra, then we have algebra morphisms  $i, j$  and coalgebra morphisms  $p, \pi$ , as in [3, Prop. 2.2], Proposition 7.1. If  $H$  is a smash cross product bialgebra, then  $\pi$  is a bialgebra morphism, and  $(A, p, j)$  can be reconstructed from  $(B, \pi, i)$ :  $(A, j)$  is the equalizer of a certain pair of morphisms, see Lemma 7.2. Conversely, if we have a bialgebra  $B$ , and a bialgebra map  $\pi : H \rightarrow B$  and an algebra map  $i : B \rightarrow H$  such that  $\pi$  is left inverse of  $i$ , then we can construct  $(A, j)$



In a similar way, we have for a morphism  $\frac{X}{Y \otimes Z}$  between  $X$  and  $Y \otimes Z$  that

$$(1.3) \quad \begin{array}{c} X \ T \\ \text{---} \\ \text{---} \\ T \ Y \ Z \end{array} = \begin{array}{c} X \ T \\ \text{---} \\ \text{---} \\ T \ Y \ Z \end{array} \quad \text{and} \quad \begin{array}{c} T \ X \\ \text{---} \\ \text{---} \\ Y \ Z \ T \end{array} = \begin{array}{c} T \ X \\ \text{---} \\ \text{---} \\ Y \ Z \ T \end{array}.$$

$c_{1,X} : \underline{1} \otimes X = X \rightarrow X \otimes \underline{1} = X$  and  $c_{X,\underline{1}} : X \otimes \underline{1} = X \rightarrow \underline{1} \otimes X = X$  are equal to

identity morphism of  $\text{Id}_X = \frac{X}{\text{---}}$ , see [7, Prop. XIII.1.2].

Let us now recall the notions of algebra and coalgebra in a monoidal category  $\mathcal{C}$ , and of bialgebra and Hopf algebra in a braided monoidal category  $\mathcal{C}$ . An algebra

in  $\mathcal{C}$  is a triple  $(A, \underline{m}_A, \underline{\eta}_A)$ , where  $A \in \mathcal{C}$ , and  $\underline{m}_A = \frac{A \ A}{A} : A \otimes A \rightarrow A$  and  $\underline{\eta}_A =$

$\frac{\underline{1}}{\underline{1}} : \underline{1} \rightarrow A$  are morphisms in  $\mathcal{C}$  satisfying the associativity and unit conditions  $\underline{m}_A \circ (\underline{m}_A \otimes \text{Id}_A) = \underline{m}_A \circ (\text{Id}_A \otimes \underline{m}_A)$  and  $\underline{m}_A \circ (\underline{\eta}_A \otimes \text{Id}_A) = \underline{m}_A \circ (\text{Id}_A \otimes \underline{\eta}_A) = \text{Id}_A$ .

A coalgebra in  $\mathcal{C}$  is a triple  $(B, \underline{\Delta}_B, \underline{\varepsilon}_B)$ , where  $B \in \mathcal{C}$ , and  $\underline{\Delta}_B = \frac{B}{B \ B} : B \rightarrow B \otimes B$

and  $\underline{\varepsilon}_B = \frac{B}{\underline{1}} : B \rightarrow \underline{1}$ , satisfying appropriate coassociativity and counit conditions.

A bialgebra in  $\mathcal{C}$  is a fivetuple  $(B, \underline{m}_B, \underline{\eta}_B, \underline{\Delta}_B, \underline{\varepsilon}_B)$ , such that  $(B, \underline{m}_B, \underline{\eta}_B)$  is an algebra and  $(B, \underline{\Delta}_B, \underline{\varepsilon}_B)$  is a coalgebra such that  $\underline{\Delta}_B : B \rightarrow B \otimes B$  and  $\underline{\varepsilon}_B : B \rightarrow \underline{1}$  are algebra morphisms.  $B \otimes B$  has the tensor product algebra structure (using the braiding on  $\mathcal{C}$ ), and  $\underline{1}$  is an algebra, with both the multiplication and unit map equal to the identity on  $\underline{1}$ . For later reference, we give explicit formulas for the axioms of a bialgebra  $B$ :  $\underline{\varepsilon}_B \underline{\eta}_B = \text{Id}_{\underline{1}}$ , and

$$(1.4) \quad \begin{array}{c} B \ B \ B \\ \text{---} \\ \text{---} \\ B \end{array} = \begin{array}{c} B \ B \ B \\ \text{---} \\ \text{---} \\ B \end{array}, \quad \begin{array}{c} B \\ \text{---} \\ \text{---} \\ B \end{array} = \begin{array}{c} B \\ \text{---} \\ \text{---} \\ B \end{array} = \begin{array}{c} B \\ \text{---} \\ \text{---} \\ B \end{array}, \quad \begin{array}{c} B \\ \text{---} \\ \text{---} \\ B \ B \ B \end{array} = \begin{array}{c} B \\ \text{---} \\ \text{---} \\ B \ B \ B \end{array},$$

$$\begin{array}{c} B \\ \text{---} \\ \text{---} \\ B \end{array} = \begin{array}{c} B \\ \text{---} \\ \text{---} \\ B \end{array} = \begin{array}{c} B \\ \text{---} \\ \text{---} \\ B \end{array}, \quad \begin{array}{c} B \ B \\ \text{---} \\ \text{---} \\ \underline{1} \end{array} = \begin{array}{c} B \ B \\ \text{---} \\ \text{---} \\ \underline{1} \end{array}, \quad \begin{array}{c} \underline{1} \\ \text{---} \\ \text{---} \\ B \ B \end{array} = \begin{array}{c} \underline{1} \\ \text{---} \\ \text{---} \\ B \ B \end{array}, \quad \begin{array}{c} B \ B \\ \text{---} \\ \text{---} \\ B \ B \end{array} = \begin{array}{c} B \ B \\ \text{---} \\ \text{---} \\ B \ B \end{array}.$$

For a bialgebra  $B$ , we can introduce the category of left  $B$ -modules  ${}_B\mathcal{C}$  and the category of left  $B$ -comodules  ${}^B\mathcal{C}$ . The left  $B$ -action on  $X \in {}_B\mathcal{C}$  is denoted by

$\frac{B \ X}{X}$ , and the left  $B$ -coaction on  $X \in {}^B\mathcal{C}$  by  $\frac{X}{B \ X}$ .  ${}_B\mathcal{C}$  and  ${}^B\mathcal{C}$  are monoidal

categories; for  $X, Y \in {}_B\mathcal{C}$  (resp.  ${}^B\mathcal{C}$ ), then  $X \otimes Y$  is a left  $B$ -module (resp. left

$B$ -comodule) via the action (resp. coaction)

$$\begin{array}{c} B \quad X \quad Y \\ \hline \text{diagram} \\ \hline X \quad Y \end{array} \left( \text{resp.} \begin{array}{c} X \quad Y \\ \hline \text{diagram} \\ \hline B \quad X \quad Y \end{array} \right).$$

A Hopf algebra in a braided monoidal category  $\mathcal{C}$  is a bialgebra  $B$  in  $\mathcal{C}$  together with a morphism  $\underline{S} : B \rightarrow B$  in  $\mathcal{C}$  (the antipode) satisfying the axioms

$$(1.5) \quad \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \end{array} = \begin{array}{c} B \\ \hline \bullet \\ \hline B \end{array} = \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \end{array}.$$

It is well-known, see [9, Lemma 2.3], that the antipode  $\underline{S}$  of a Hopf algebra  $B$  in a braided monoidal category  $\mathcal{C}$  is an anti-algebra and anti-coalgebra morphism, in the sense that

$$(1.6) \quad (a) \quad \begin{array}{c} B \quad B \\ \hline \text{diagram} \\ \hline B \end{array} = \begin{array}{c} B \quad B \\ \hline \text{diagram} \\ \hline B \end{array}, \quad \begin{array}{c} 1 \\ \hline \bullet \\ \hline B \end{array} = \begin{array}{c} 1 \\ \hline \bullet \\ \hline B \end{array} \quad \text{and} \quad (b) \quad \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \quad B \end{array} = \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \quad B \end{array}, \quad \begin{array}{c} B \\ \hline \bullet \\ \hline 1 \end{array} = \begin{array}{c} B \\ \hline \bullet \\ \hline 1 \end{array}.$$

## 2. CROSS PRODUCT ALGEBRAS AND COALGEBRAS

Let  $A$  and  $B$  be algebras and coalgebras in  $\mathcal{C}$ , but not necessarily bialgebras. Consider morphisms  $\psi = \frac{B \quad A}{A \quad B} : B \otimes A \rightarrow A \otimes B$  and  $\phi = \frac{A \quad B}{B \quad A} : A \otimes B \rightarrow B \otimes A$  in  $\mathcal{C}$ .  $\psi$  and  $\phi$  can be used to define a multiplication and a comultiplication on  $A \otimes B$ :

$$\begin{array}{c} A \quad B \quad A \quad B \\ \hline \text{diagram} \\ \hline A \quad B \end{array} ; \quad \begin{array}{c} A \quad B \\ \hline \text{diagram} \\ \hline A \quad B \quad A \quad B \end{array}.$$

$A \#_{\psi} B$  is  $A \otimes B$  together with the multiplication induced by  $\psi$ , and with unit map  $\underline{\eta}_A \otimes \underline{\eta}_B$ ;  $A \#^{\phi} B$  is  $A \otimes B$  together with the comultiplication induced by  $\phi$ , and with counit map  $\underline{\varepsilon}_A \otimes \underline{\varepsilon}_B$ . If  $A \#_{\psi} B$  is an algebra in  $\mathcal{C}$ , then we say that  $A \#_{\psi} B$  is a cross product algebra of  $A$  and  $B$ ; if  $A \#^{\phi} B$  is a coalgebra in  $\mathcal{C}$ , then we say that  $A \#^{\phi} B$  is a cross product coalgebra of  $A$  and  $B$ .  $A \otimes B$  together with the multiplication induced by  $\psi$ , the comultiplication induced by  $\phi$ , unit  $\underline{\eta}_A \otimes \underline{\eta}_B$  and counit  $\underline{\varepsilon}_A \otimes \underline{\varepsilon}_B$  will be denoted by  $A \#^{\phi}_{\psi} B$ . If  $A \#^{\phi}_{\psi} B$  is a bialgebra in  $\mathcal{C}$ , then we call it a cross product bialgebra.

It is known, see for example [6, Theorem 2.5] in the case where  $\mathcal{C}$  is the category of vector spaces, that  $A \#_{\psi} B$  is a cross product algebra if and only the four following relations hold:

$$(2.1) \quad (a) \quad \begin{array}{c} B \quad B \quad A \\ \hline \text{diagram} \\ \hline A \quad B \end{array} = \begin{array}{c} B \quad B \quad A \\ \hline \text{diagram} \\ \hline A \quad B \end{array}, \quad (b) \quad \begin{array}{c} B \quad A \quad A \\ \hline \text{diagram} \\ \hline A \quad B \end{array} = \begin{array}{c} B \quad A \quad A \\ \hline \text{diagram} \\ \hline A \quad B \end{array}$$

$$(c) \quad \begin{array}{c} B \\ \hline \text{---} \end{array} = \begin{array}{c} B \\ \hline \text{---} \end{array}, \quad (d) \quad \begin{array}{c} A \\ \hline \text{---} \end{array} = \begin{array}{c} A \\ \hline \text{---} \end{array}.$$

This can be restated in the language of monoidal categories. For an algebra  $A$  in  $\mathcal{C}$ , we consider the category  $\mathfrak{T}_A$  of right transfer morphisms through  $A$ . The objects are pairs  $(X, \psi_{X,A})$  with  $X \in \mathcal{C}$  and  $\psi_{X,A} : X \otimes A \rightarrow A \otimes X$  a morphism in  $\mathcal{C}$  such that

$$\begin{array}{c} X \ A \ A \\ \hline \text{---} \end{array} = \begin{array}{c} X \ A \ A \\ \hline \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} X \\ \hline \text{---} \end{array} = \begin{array}{c} X \\ \hline \text{---} \end{array}.$$

A morphism in  $\mathfrak{T}_A$  between  $(X, \psi_{X,A})$  and  $(Y, \psi_{Y,A})$  is a morphism  $\mu : X \rightarrow Y$  in  $\mathcal{C}$  such that  $(\text{Id}_A \otimes \mu) \circ \psi_{X,A} = \psi_{Y,A} \circ (\mu \otimes \text{Id}_A)$ .  $\mathfrak{T}_A$  is a strict monoidal category, with unit object  $(\underline{1}, \text{Id}_A)$  and tensor product

$$(X, \psi_{X,A}) \otimes (Y, \psi_{Y,A}) = (X \otimes Y, \psi_{X \otimes Y, A}), \quad \text{with } \psi_{X \otimes Y, A} := \begin{array}{c} X \ Y \ A \\ \hline \text{---} \end{array}.$$

The category  ${}^A\mathfrak{T}$  of left transfer morphisms through  $A$  is defined in a similar way, and is also a strict monoidal category. Then we have the following result, going back to [16], see also [14, Sec. 4].

**Proposition 2.1.** *Let  $A$  and  $B$  be algebras in a strict monoidal category  $\mathcal{C}$ , and  $\psi : B \otimes A \rightarrow A \otimes B$  a morphism in  $\mathcal{C}$ . Then the following assertions are equivalent:*

- (i)  $A \#_{\psi} B$  is a cross product algebra;
- (ii)  $(B, \psi)$  is an algebra in  $\mathfrak{T}_A$ ;
- (iii)  $(A, \psi)$  is an algebra in  ${}_B\mathfrak{T}$ .

*Proof.* Observe  $(B, \psi) \in \mathfrak{T}_A$  is equivalent to (2.1.b-c); if these hold, then (2.1.a) and (2.1.d) mean precisely that  $(B, \psi)$  is an algebra in  $\mathfrak{T}_A$ . This proves the equivalence of (i) and (ii). The equivalence of (i) and (iii) can be proved in a similar way: (2.1.a) and (2.1.d) are equivalent to  $(A, \psi) \in {}_B\mathfrak{T}$ , and then the two other conditions mean that  $(A, \psi)$  is an algebra in  ${}_B\mathfrak{T}$ .  $\square$

Recall that an augmented algebra is a pair  $(B, \underline{\varepsilon}_B)$ , where  $B$  is an algebra, and  $\underline{\varepsilon}_B : B \rightarrow \underline{1}$  is an algebra morphism.

**Lemma 2.2.** *Let  $A \#_{\psi} B$  be a cross product algebra. If  $(B, \underline{\varepsilon}_B)$  is an augmented algebra then  $A \in {}_B\mathcal{C}$  via (2.2.a). If  $(A, \underline{\varepsilon}_A)$  is an augmented algebra then  $B \in \mathcal{C}_A$  via (2.2.b).*

$$(2.2) \quad (a) \quad \begin{array}{c} B \ A \\ \hline \text{---} \end{array} := \begin{array}{c} B \ A \\ \hline \text{---} \end{array}; \quad (b) \quad \begin{array}{c} B \ A \\ \hline \text{---} \end{array} := \begin{array}{c} B \ A \\ \hline \text{---} \end{array}.$$

*Proof.* Composing (2.1.a) and (2.1.d) to the left with  $\text{Id}_A \otimes \underline{\varepsilon}_B$ , we find that  $A$  is a left  $B$ -module.  $\square$

For further reference, we record the dual results. We leave it to the reader to introduce the monoidal categories  ${}^A\mathfrak{T}$  and  $\mathfrak{T}^A$  of left and right transfer morphisms through the coalgebra  $A$ .

**Proposition 2.3.** *Let  $A$  and  $B$  be coalgebras, and let  $\phi : A \otimes B \rightarrow B \otimes A$  be a morphism in  $\mathcal{C}$ . Then the following statements are equivalent:*

1)  $A \#^\phi B$  is a cross product coalgebra;

2) the following relations hold:

$$(2.3) \quad \begin{aligned} (a) \quad & \begin{array}{c} A \ B \\ \hline \text{Diagram 1} \end{array} = \begin{array}{c} A \ B \\ \hline \text{Diagram 2} \end{array}, \quad (b) \quad \begin{array}{c} A \ B \\ \hline \text{Diagram 3} \end{array} = \begin{array}{c} A \ B \\ \hline \text{Diagram 4} \end{array}, \\ (c) \quad & \begin{array}{c} A \ B \\ \hline \text{Diagram 5} \end{array} = \begin{array}{c} A \ B \\ \hline \text{Diagram 6} \end{array}, \quad (d) \quad \begin{array}{c} A \ B \\ \hline \text{Diagram 7} \end{array} = \begin{array}{c} A \ B \\ \hline \text{Diagram 8} \end{array}; \end{aligned}$$

3)  $(B, \phi)$  is a coalgebra in  ${}^A\mathfrak{T}$ ;

4)  $(A, \phi)$  is a coalgebra in  $\mathfrak{T}^B$ .

A coaugmented coalgebra is a pair  $(B, \underline{\eta}_B)$ , where  $B$  is a coalgebra, and  $\underline{\eta}_B : \underline{1} \rightarrow B$  is a coalgebra morphism.

**Lemma 2.4.** *Assume that  $A \#^\phi B$  is a cross product coalgebra. If  $(B, \underline{\eta}_B)$  is a coaugmented coalgebra, then  $A \in {}^B\mathcal{C}$  via (2.4.a). If  $(A, \underline{\eta}_A)$  is a coaugmented coalgebra, then  $B \in \mathcal{C}^A$  via (2.4.b).*

$$(2.4) \quad \begin{aligned} (a) \quad & \begin{array}{c} A \\ \hline \text{Diagram 9} \end{array} := \begin{array}{c} A \\ \hline \text{Diagram 10} \end{array}; \quad (b) \quad \begin{array}{c} B \\ \hline \text{Diagram 11} \end{array} := \begin{array}{c} A \\ \hline \text{Diagram 12} \end{array}. \end{aligned}$$

### 3. SMASH PRODUCT ALGEBRAS AND COALGEBRAS

These are particular examples of cross product algebras and coalgebras. Assume that  $B$  is a bialgebra, so that  ${}_B\mathcal{C}$ , the category of left  $B$ -representations, and  ${}^B\mathcal{C}$ , the category of left  $B$ -corepresentations, are monoidal categories.

For an algebra  $A$  in  ${}_B\mathcal{C}$ , we have a cross product algebra  $A \#_\psi B$ , with  $\psi = \begin{array}{c} B \ A \\ \hline \text{Diagram 13} \\ \hline A \ B \end{array}$ ,

where the left  $B$ -action on  $A$  is  $\begin{array}{c} B \ A \\ \hline \text{Diagram 14} \\ \hline A \end{array}$ . This algebra is called the left smash product algebra of  $A$  and  $B$ .

In a similar way, for a coalgebra  $A$  in  ${}_B\mathcal{C}$ , we have a cross product coalgebra  $A \#^\phi B$ ,

with  $\phi = \begin{array}{c} A \ B \\ \hline \text{Diagram 15} \\ \hline B \ A \end{array}$ , where the left  $B$ -coaction on  $A$  is  $\begin{array}{c} A \\ \hline \text{Diagram 16} \\ \hline B \ A \end{array}$ . This coalgebra is called a left smash product coalgebra.

We remark that right smash product algebras and coalgebras can be considered as well.

Assume that  $B$  is a bialgebra, and that  $A \#_{\psi} B$  is a cross product algebra. In Proposition 3.1, we discuss when  $A \#_{\psi} B$  is a smash product algebra.

**Proposition 3.1.** *Let  $B$  be a bialgebra, and let  $A$  be an algebra, and consider  $\psi : B \otimes A \rightarrow A \otimes B$  such that  $A \#_{\psi} B$  is a cross product algebra.  $A \#_{\psi} B$  is a left smash product algebra if and only if*

$$(3.1) \quad \psi = \begin{array}{c} \begin{array}{cc} B & A \end{array} \\ \hline \begin{array}{c} \text{diagram of a braid with two strands, one labeled B and one labeled A, with a crossing and a dot on the B strand} \end{array} \\ \hline \begin{array}{cc} A & B \end{array} \end{array}.$$

Moreover, the full subcategory  ${}_B\mathfrak{T}'$  of  ${}_B\mathfrak{T}$ , with objects of the form  $(X, \psi)$ , where  $\psi$  satisfies (3.1), with  $A$  replaced by  $X$ , is a monoidal subcategory of  ${}_B\mathfrak{T}$  that is monoidal isomorphic to  ${}_B\mathcal{C}$ .

*Proof.* Assume first that  $A \#_{\psi} B$  is a smash product algebra. Then  $A$  is an algebra in  ${}_B\mathcal{C}$  and

$$(3.2) \quad \begin{array}{c} \text{Diagram 1} \\ \hline A \quad B \quad A \quad B \\ \hline \end{array} = \begin{array}{c} \text{Diagram 2} \\ \hline A \quad B \quad A \quad B \\ \hline \end{array}$$

Composing (3.2) to the right with  $\underline{\eta}_A \otimes \text{Id}_{B \otimes A} \otimes \underline{\eta}_B$ , we obtain that

$$\frac{B \ A}{A \ B} = \frac{B \ A}{A \ B} \text{ , hence } \frac{B \ A}{A \ B} = \frac{B \ A}{A} \text{ ,}$$

and this implies (3.1).

Conversely, assume that  $A \#_\psi B$  is a cross product algebra and that  $\psi$  satisfies (3.1).

We know that  $A \in {}_B\mathcal{C}$ , with left  $B$ -action (2.2.a).  $A$  is an algebra in  ${}_B\mathcal{C}$  since

The multiplication on the smash product algebra is

$$\begin{array}{c} A \quad B \quad A \quad B \\ \hline \text{Diagram 1} \\ \hline A \quad B \end{array} = \begin{array}{c} A \quad B \quad A \quad B \\ \hline \text{Diagram 2} \\ \hline A \quad B \end{array} \stackrel{(3.1)}{=} \begin{array}{c} A \quad B \quad A \quad B \\ \hline \text{Diagram 3} \\ \hline A \quad B \end{array}$$



and coincides with the multiplication on the cross product algebra  $A \#_\psi B$ . This finishes the proof of the first statement.

We next show that  ${}_B\mathfrak{T}'$  is closed under the tensor product: if  $(X, \psi_{B,X}), (Y, \psi_{B,Y}) \in {}_B\mathfrak{T}'$ , then  $(X, \psi_{B,X}) \underline{\otimes} (Y, \psi_{B,Y}) \in {}_B\mathfrak{T}'$ , since

$$\begin{array}{c} \text{Diagram 1} \end{array} \stackrel{(3.1)}{=} \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(1.3)}{=} \begin{array}{c} \text{Diagram 4} \end{array} \stackrel{(3.1)}{=} \begin{array}{c} \text{Diagram 5} \end{array}.$$

Finally, we will construct a monoidal isomorphism  $F : {}_B\mathfrak{T}' \rightarrow {}_B\mathcal{C}$ . Take  $(X, \psi) \in {}_B\mathfrak{T}'$ . In the first part of the proof, we have seen that  $X \in {}_B\mathcal{C}$  via the  $B$ -action

$$\frac{B \ X}{X} = \frac{B \ X}{X} \quad \text{and this defines } F \text{ at the level of objects. At the level of morphisms,}$$

$F$  acts as the identity. Now we define a functor  $G : {}_B\mathcal{C} \rightarrow {}_B\mathfrak{T}'$ . Take a left  $B$ -

$$\text{module } X, \text{ and let } \psi = \frac{B \ X}{X \ B} =: \frac{B \ X}{X \ B}. \text{ Then } \frac{B \ X}{X} = \frac{B \ X}{X}, \text{ and therefore } \psi$$

satisfies (3.1).  $(X, \psi_{B,X})$  is an object of  ${}_B\mathfrak{T}$  since

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} \stackrel{(1.2)}{=} \begin{array}{c} \text{Diagram 4} \end{array} \stackrel{(1.2)}{=} \begin{array}{c} \text{Diagram 5} \end{array}.$$

and

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array}.$$

We conclude that  $(X, \psi_{B,X}) \in {}_B\mathfrak{T}'$ , and we define  $G(X) = (X, \psi_{B,X})$ . At the level of morphisms,  $G$  acts as the identity. Using (3.1), we can show that  $F$  and  $G$  are inverses. Finally, using the coassociativity of the comultiplication on  $B$  and (1.3),

we can prove that

and this implies that  $F$  is a strictly monoidal functor.  $\square$

We end this Section with the dual version of Proposition 3.1. Verification of the details is left to the reader.

**Proposition 3.2.** *Let  $B$  be a bialgebra, and let  $A$  be a coalgebra. Assume that  $\phi : A \otimes B \rightarrow B \otimes A$  is such that  $A \#^\phi B$  is a cross product coalgebra.  $A \#^\phi B$  is a smash product coalgebra if and only if*

$$(3.3) \quad \phi = \text{[Diagram: A box with two inputs labeled A and B at the top, and two outputs labeled B and A at the bottom. Inside the box, the lines cross and there is a dot on the line from A to B].}$$

The full subcategory of  $\mathfrak{T}^B$  consisting of objects  $(X, \phi)$  satisfying (3.3), with  $A$  replaced by  $X$ , is strictly monoidal and can be identified to  $\mathcal{C}^B$  as a monoidal category.

#### 4. CROSS PRODUCT BIALGEBRAS

Suppose that  $A$  and  $B$  are algebras and coalgebras, and that we have morphisms  $\psi : B \otimes A \rightarrow A \otimes B$  and  $\phi : A \otimes B \rightarrow B \otimes A$  such that  $A \#_\psi B$  is a cross product algebra and  $A \#^\phi B$  is a cross product coalgebra. Then we will call  $(A, B, \psi, \phi)$  a *cross product algebra-coalgebra datum*. In [3, Sec. 2],  $(A, B, \psi, \phi)$  is called a *bialgebra admissible tuple*, or a BAT, if  $A \#^\phi B$  is a cross product bialgebra. Take a cross product algebra-coalgebra datum  $(A, B, \psi, \phi)$ . We will produce a list of properties that are satisfied if  $(A, B, \psi, \phi)$  is an admissible tuple; otherwise stated, we will make a list of necessary conditions for  $A \#^\phi B$  being a cross product bialgebra. Then we will identify subsets of this list of properties that guarantee that  $(A, B, \psi, \phi)$  is a bialgebra admissible tuple, in other words, sets of necessary and sufficient conditions for  $A \#^\phi B$  being a cross product bialgebra. The results will be summarized in Theorem 4.6.

$A \#^\phi B$  is a cross product bialgebra if and only if the comultiplication and counit are algebra maps; these conditions come down to the following equalities:

$$(a) \quad \text{[Diagram: A box with four inputs labeled A, B, A, B at the top and four outputs labeled A, B, A, B at the bottom. Inside, the lines are connected in a specific pattern representing the comultiplication map.]}$$

$$(b) \quad \text{[Diagram: A box with four inputs labeled A, B, A, B at the bottom and two outputs labeled 1 at the top. Inside, the lines are connected in a specific pattern representing the counit map.]}$$

(4.4)

$$\begin{aligned}
\text{(e)} \quad & \begin{array}{c} \text{A B A} \\ \hline \text{A A} \end{array} = \begin{array}{c} \text{A B A} \\ \hline \text{A A} \end{array}, \quad \text{(f)} \quad \begin{array}{c} \text{B A B} \\ \hline \text{B B} \end{array} = \begin{array}{c} \text{B A B} \\ \hline \text{B B} \end{array}, \\
\text{(g)} \quad & \begin{array}{c} \text{B A} \\ \hline \text{A B} \end{array} = \begin{array}{c} \text{B A} \\ \hline \text{A B} \end{array} \quad \text{and} \quad \text{(h)} \quad \begin{array}{c} \text{A B A B} \\ \hline \text{B A} \end{array} = \begin{array}{c} \text{A B A B} \\ \hline \text{B A} \end{array}.
\end{aligned}$$

*Proof.* Compose (4.1.a) to the right with  $\text{Id}_A \otimes \underline{\eta}_B \otimes \text{Id}_{A \otimes B}$  and to the left with  $\text{Id}_{A \otimes B \otimes A} \otimes \underline{\varepsilon}_B$ . Applying (2.1.d), (4.2.b) and (4.3.b), and the fact that  $\underline{\varepsilon}_B \underline{\eta}_B = \text{Id}_1$  we obtain

$$(4.5) \quad \begin{array}{c} \text{A A B} \\ \hline \text{A B A} \end{array} = \begin{array}{c} \text{A A B} \\ \hline \text{A B A} \end{array}.$$

We find (4.4.a) after we compose (4.5) to the right with  $\text{Id}_{A \otimes A} \otimes \underline{\eta}_B$ . Composing (4.5) to the right with  $\text{Id}_A \otimes \underline{\eta}_A \otimes \text{Id}_B$  and to the left with  $\underline{\varepsilon}_A \otimes \text{Id}_{B \otimes A}$ , and with the help of (4.2.a) and (2.1.c), we deduce (4.4.c).

Now compose (4.1.a) to the right with  $\text{Id}_{A \otimes B} \otimes \underline{\eta}_A \otimes \text{Id}_B$  and to the left with  $\underline{\varepsilon}_A \otimes \text{Id}_{B \otimes A \otimes B}$ . Combining the resulting equation with (2.1.c), we obtain that

$$(4.6) \quad \begin{array}{c} \text{A B B} \\ \hline \text{B A B} \end{array} = \begin{array}{c} \text{A B B} \\ \hline \text{B A B} \end{array}.$$

After we compose (4.6) to the right with  $\underline{\eta}_A \otimes \text{Id}_{B \otimes B}$ , we obtain (4.4.b).

Compose (4.1.a) to the right with  $\text{Id}_{A \otimes B \otimes A} \otimes \underline{\eta}_B$  and to the left with  $\text{Id}_A \otimes \underline{\varepsilon}_B \otimes \text{Id}_{A \otimes B}$ . Then by (2.3.d), we obtain that

$$(4.7) \quad \begin{array}{c} \text{A B A} \\ \hline \text{A A B} \end{array} = \begin{array}{c} \text{A B A} \\ \hline \text{A A B} \end{array}.$$

Composing (4.7) to the right with  $\underline{\eta}_A \otimes \text{Id}_{B \otimes A}$  and to the left with  $\text{Id}_A \otimes \underline{\varepsilon}_A \otimes \text{Id}_B$ , and using (2.3.c) and (4.3.a), we find (4.4.g). Composing (4.7) to the left with  $\text{Id}_{A \otimes A} \otimes \underline{\varepsilon}_B$ , we find (4.4.e).

Now compose (4.1.a) to the left with  $\text{Id}_{A \otimes B} \otimes \underline{\varepsilon}_A \otimes \text{Id}_B$  and to the right with  $\underline{\eta}_A \otimes \text{Id}_{B \otimes A \otimes B}$ . This gives

$$(4.8) \quad \begin{array}{c} \text{B A B} \\ \text{A B B} \end{array} = \begin{array}{c} \text{B A B} \\ \text{A B B} \end{array}.$$

Composing (4.8) to the left with  $\underline{\varepsilon}_A \otimes \text{Id}_{B \otimes B}$ , we obtain (4.4.f).

(4.4.d) follows after we compose (4.1.a) to the right with  $\underline{\eta}_A \otimes \text{Id}_{B \otimes A} \otimes \underline{\eta}_B$ , and then use (4.2.a-b). Finally, (4.4.h) follows after we compose (4.1.a) to the left with  $\underline{\varepsilon}_A \otimes \text{Id}_{B \otimes A} \otimes \underline{\varepsilon}_B$ , and then use (4.3.a-b).  $\square$

Observe that we could have skipped half of the proof: (4.4.e-h) follow from (4.4.a-d) using duality arguments.

Applying Proposition 4.1, we find some more properties of bialgebra admissible tuples. They deserve a separate formulation for two reasons: they appear also in the Bespalov-Drabant list, and they play a key role in the formulation of Theorem 4.6.

**Corollary 4.2.** *If  $A \#_\psi^\phi B$  is a cross product bialgebra then the following equalities hold:*

$$(4.9) \quad \begin{array}{l} \text{(a)} \quad \begin{array}{c} \text{A A} \\ \text{A A} \end{array} = \begin{array}{c} \text{A A} \\ \text{A A} \end{array}, \quad \text{(b)} \quad \begin{array}{c} \text{B B} \\ \text{B B} \end{array} = \begin{array}{c} \text{B B} \\ \text{B B} \end{array}, \\ \text{(c)} \quad \begin{array}{c} \text{A A} \\ \text{B A} \end{array} = \begin{array}{c} \text{A A} \\ \text{B A} \end{array}, \quad \text{(d)} \quad \begin{array}{c} \text{B B} \\ \text{B A} \end{array} = \begin{array}{c} \text{B B} \\ \text{B A} \end{array}, \\ \text{(e)} \quad \begin{array}{c} \text{B A} \\ \text{A A} \end{array} = \begin{array}{c} \text{B A} \\ \text{A A} \end{array} \quad \text{and} \quad \text{(f)} \quad \begin{array}{c} \text{B A} \\ \text{B B} \end{array} = \begin{array}{c} \text{B A} \\ \text{B B} \end{array}. \end{array}$$

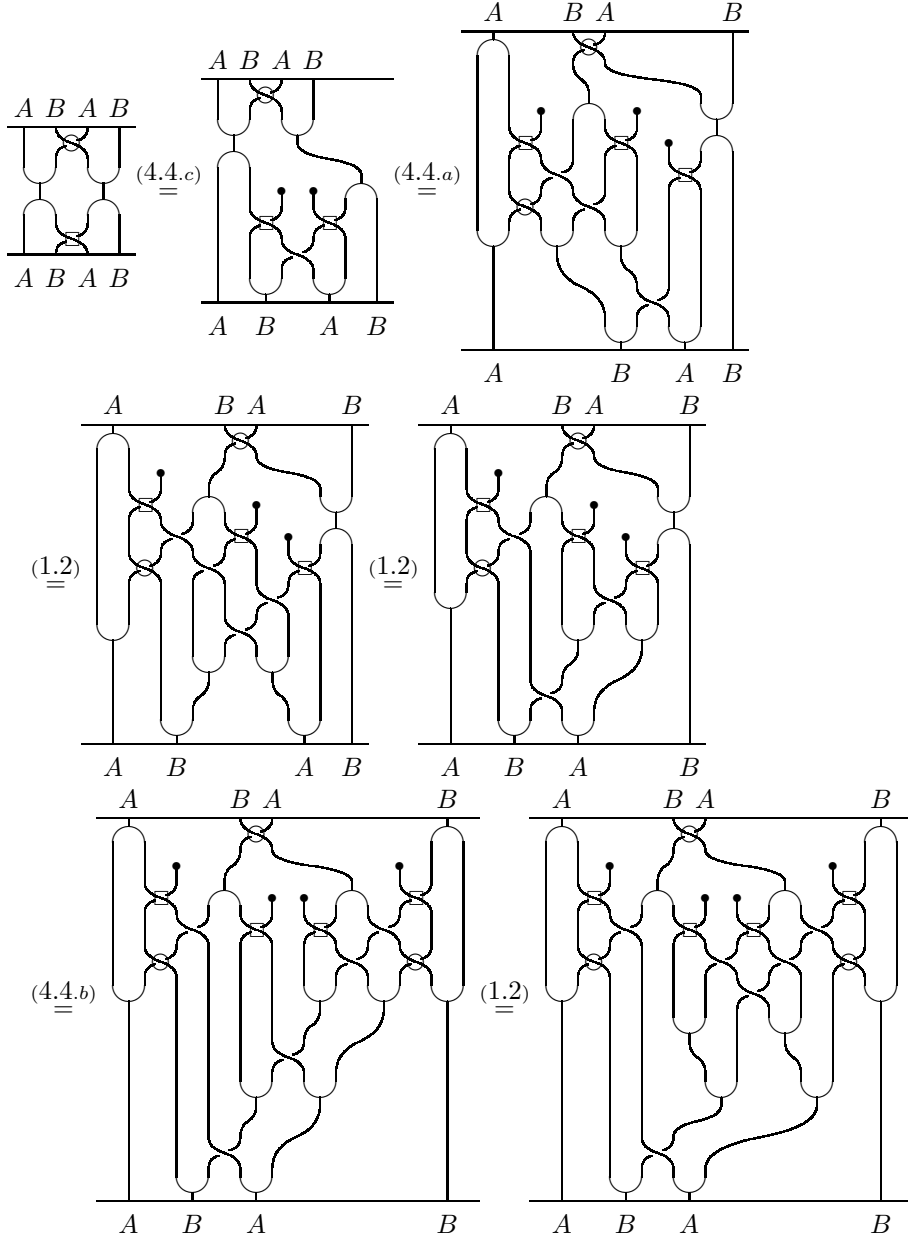
(algebra-coalgebra compatibility) (comodule-algebra compatibility) (module-coalgebra compatibility)

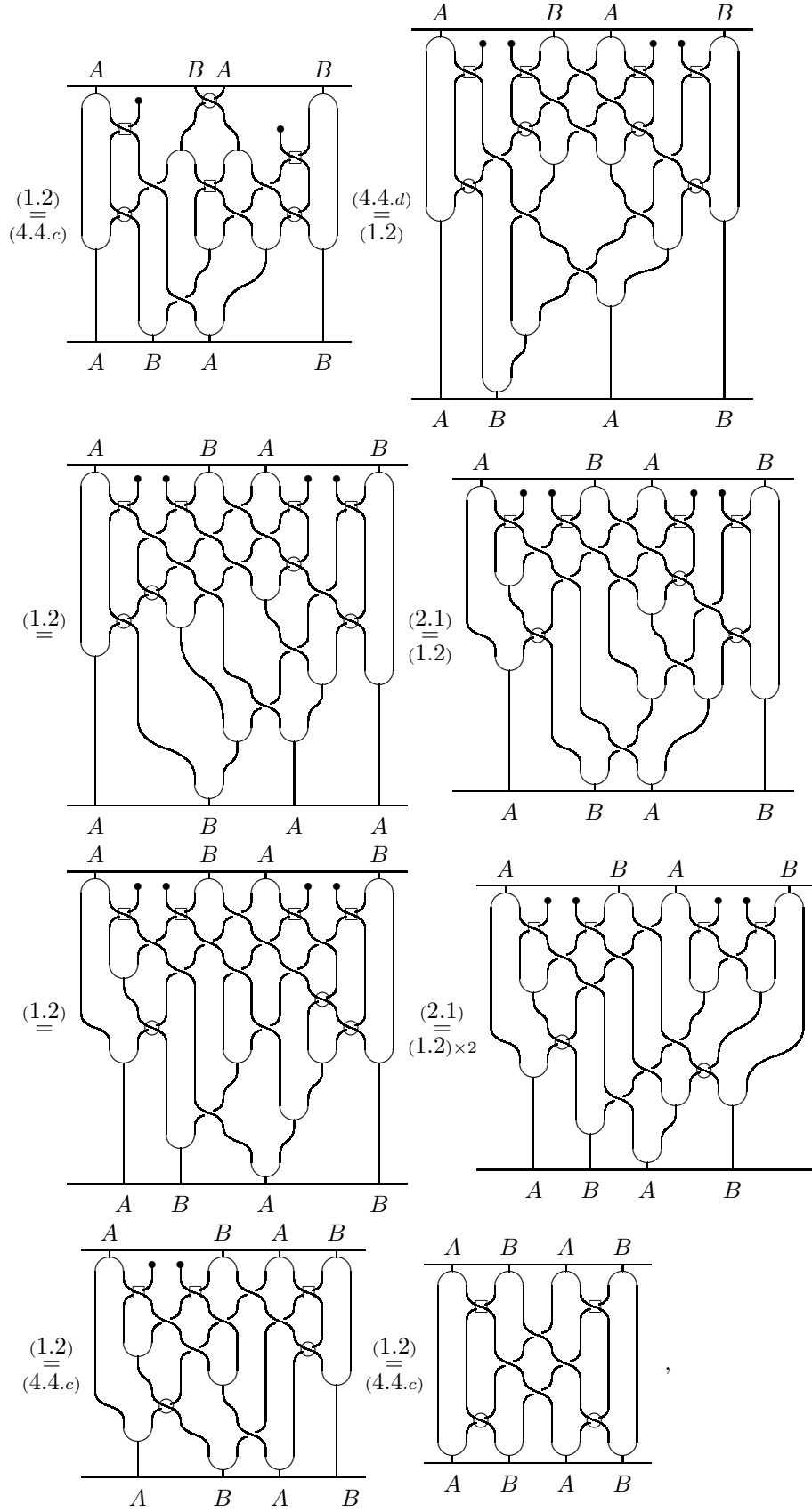
*Proof.* (4.9.a) follows after we compose (4.4.a) to the left with  $\text{Id}_A \otimes \underline{\varepsilon}_B \otimes \text{Id}_A$ , and (4.9.b) follows after we compose (4.4.b) to the left with  $\text{Id}_B \otimes \underline{\varepsilon}_A \otimes \text{Id}_B$ . In a similar way, (4.9.c) follows after we compose (4.4.a) to the left with  $\underline{\varepsilon}_A \otimes \text{Id}_{B \otimes A}$ , and (4.9.d) follows after we compose (4.4.b) to the left with  $\text{Id}_{B \otimes A} \otimes \underline{\varepsilon}_B$ . Finally,

(4.9.e) follows after we compose (4.4.d) to the left with  $\text{Id}_A \otimes \underline{\varepsilon}_B \otimes \text{Id}_A \otimes \underline{\varepsilon}_B$ , and (4.9.f) follows after we compose (4.4.d) to the left with  $\underline{\varepsilon}_A \otimes \text{Id}_B \otimes \underline{\varepsilon}_A \otimes \text{Id}_B$ . Note that in all these computations, we have to use freely the relations (4.2) and (4.3), and the fact that  $\underline{\varepsilon}_A \underline{\eta}_A = \text{Id}_1 = \underline{\varepsilon}_B \underline{\eta}_B$ .  $\square$

**Proposition 4.3.** *Let  $(A, B, \psi, \phi)$  be a crossed product algebra-coalgebra datum. Assume that (4.4.a-d) or (4.4.e-h) holds. Then (4.1.a) holds, that is, the comultiplication on  $A \#^\phi_\psi B$  is multiplicative.*

*Proof.* We only prove the first assertion; the proof of the second one is similar, and can also be obtained by duality arguments. The first assertion follows from the following computation





as desired. At some steps, we used the associativity of  $\underline{m}_A$  and  $\underline{m}_B$ , and the naturality of the braiding.  $\square$

**Corollary 4.4.** *Let  $(A, B, \psi, \phi)$  be a cross product algebra-coalgebra datum. Then the following assertions are equivalent:*

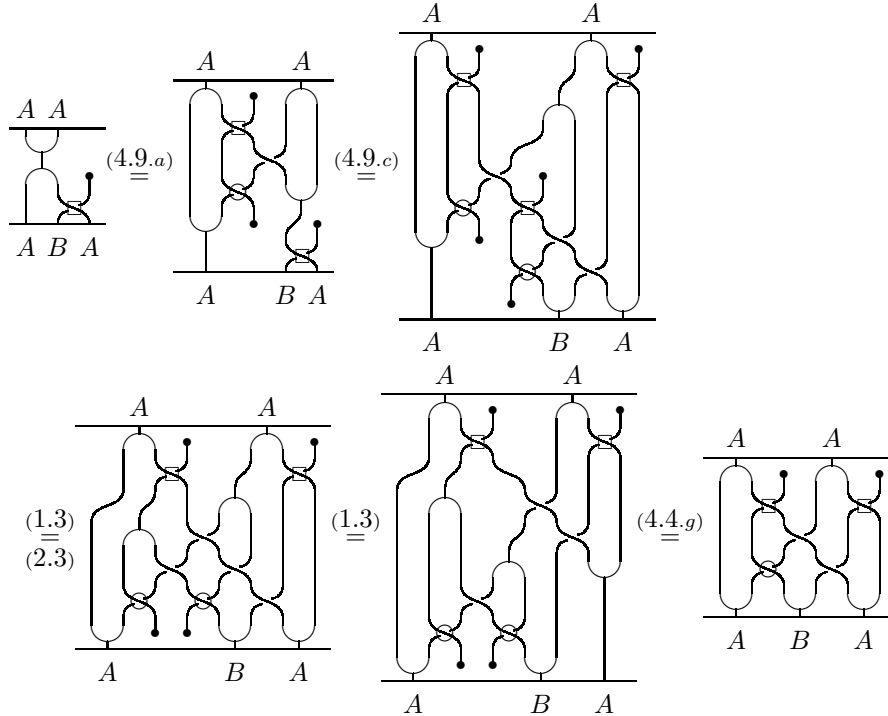
- (i)  $(A, B, \psi, \phi)$  is a bialgebra admissible tuple, that is,  $A \#^\phi_\psi B$  is a cross product bialgebra;
- (ii)  $\underline{\varepsilon}_X \underline{\eta}_X = \text{Id}_1$  for  $X \in \{A, B\}$ , and (4.2), (4.3) and (4.4.a-d) hold;
- (iii)  $\underline{\varepsilon}_X \underline{\eta}_X = \text{Id}_1$  for  $X \in \{A, B\}$ , and (4.2), (4.3) and (4.4.e-h) hold.

Corollary 4.4 is a first list of sets of necessary and sufficient conditions for a cross product algebra-coalgebra datum being a bialgebra admissible tuple. Before we can extend this list, we need another Lemma.

**Lemma 4.5.** *Let  $(A, B, \psi, \phi)$  be a cross product algebra-coalgebra datum and assume that  $\underline{\varepsilon}_X \underline{\eta}_X = \text{Id}_1$ , for  $X \in \{A, B\}$ , and that (4.2-4.3) hold.*

- (i) *If (4.4.g) holds then (4.4.a) is equivalent to (4.9.a,c), and (4.4.b) is equivalent to (4.9.b,d).*
- (ii) *If (4.4.c) holds then (4.4.e) is equivalent to (4.9.a,e), and (4.4.f) is equivalent to (4.9.b,f).*

*Proof.* We will prove the first statement of (i), the proof of all the other assertions is similar. If (4.4.a) holds, then (4.9.a), resp. (4.9.c) follows after we compose (4.4.a) to the left with  $\text{Id}_A \otimes \underline{\varepsilon}_B \otimes \text{Id}_A$ , resp.  $\underline{\varepsilon}_A \otimes \text{Id}_{B \otimes A}$ , see the proof of Corollary 4.2. The proof of the converse implication follows from our next computation:



$\square$



Theorem 4.6 is the main result of this Section. Two new conditions will appear, namely

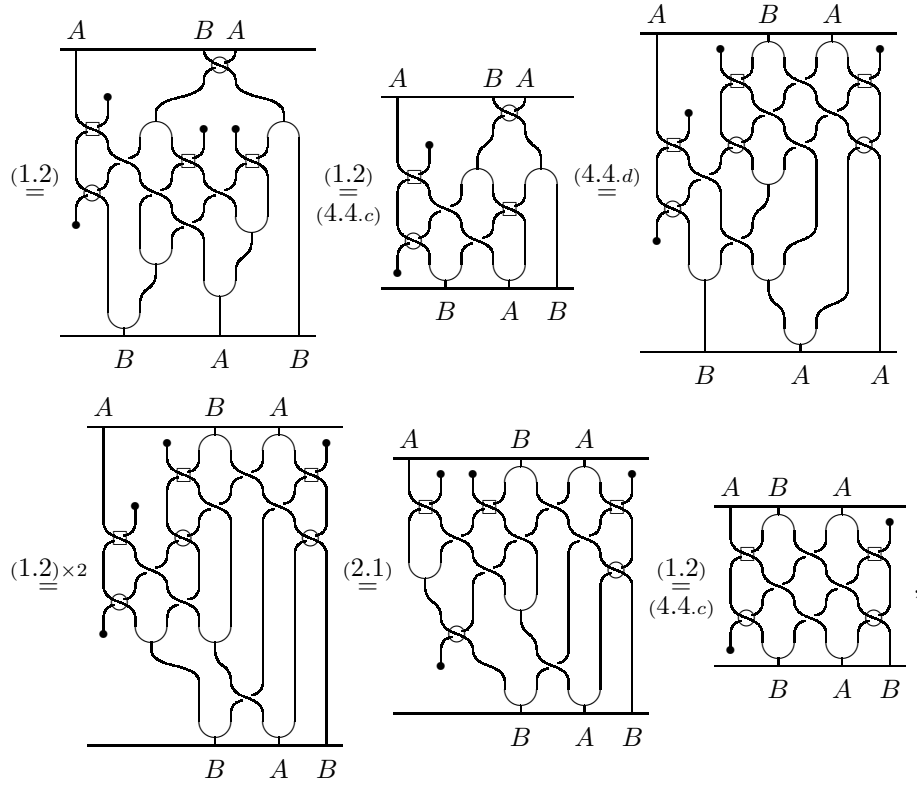
$$(4.10) \quad (a) \quad \begin{array}{c} A \ B \ A \\ \text{---} \\ \text{[Diagram: A box with two crossings, top-left to bottom-right and top-right to bottom-left]} \\ \text{---} \\ B \ A \ B \end{array} = \begin{array}{c} A \ B \ A \\ \text{---} \\ \text{[Diagram: A box with two crossings, top-left to bottom-right and top-right to bottom-left]} \\ \text{---} \\ B \ A \ B \end{array} \quad \text{or} \quad (b) \quad \begin{array}{c} B \ A \ B \\ \text{---} \\ \text{[Diagram: A box with two crossings, top-left to bottom-right and top-right to bottom-left]} \\ \text{---} \\ A \ B \ A \end{array} = \begin{array}{c} B \ A \ B \\ \text{---} \\ \text{[Diagram: A box with two crossings, top-left to bottom-right and top-right to bottom-left]} \\ \text{---} \\ A \ B \ A \end{array}.$$

**Theorem 4.6.** *Let  $(A, B, \psi, \phi)$  be a cross product algebra-coalgebra datum. Then the following assertions are equivalent,*

- (i)  $A \#_{\psi}^{\phi} B$  is a cross product bialgebra;
- (ii)  $\varepsilon_X \eta_X = \text{Id}_1$  for  $X \in \{A, B\}$ , and (4.2), (4.3) and (4.4.a-d) hold;
- (iii)  $\underline{\varepsilon}_X \eta_X = \text{Id}_1$  for  $X \in \{A, B\}$ , and (4.2), (4.3) and (4.4.e-h) hold;
- (iv)  $\underline{\varepsilon}_X \eta_X = \text{Id}_1$  for  $X \in \{A, B\}$ , and (4.2), (4.3), (4.4.c,d,g) and (4.9.a-d) hold;
- (v)  $\underline{\varepsilon}_X \eta_X = \text{Id}_1$  for  $X \in \{A, B\}$ , and (4.2), (4.3), (4.4.c,g,h) and (4.9.a,b,e,f) hold;
- (vi)  $\underline{\varepsilon}_X \eta_X = \text{Id}_1$  for  $X \in \{A, B\}$ , and (4.2), (4.3), (4.4.c,g), (4.10.a) and (4.9.a,b,d,e) hold;
- (vii)  $\underline{\varepsilon}_X \eta_X = \text{Id}_1$  for  $X \in \{A, B\}$ , and (4.2), (4.3), (4.4.c,g), (4.10.b) and (4.9.a,b,c,f) hold.

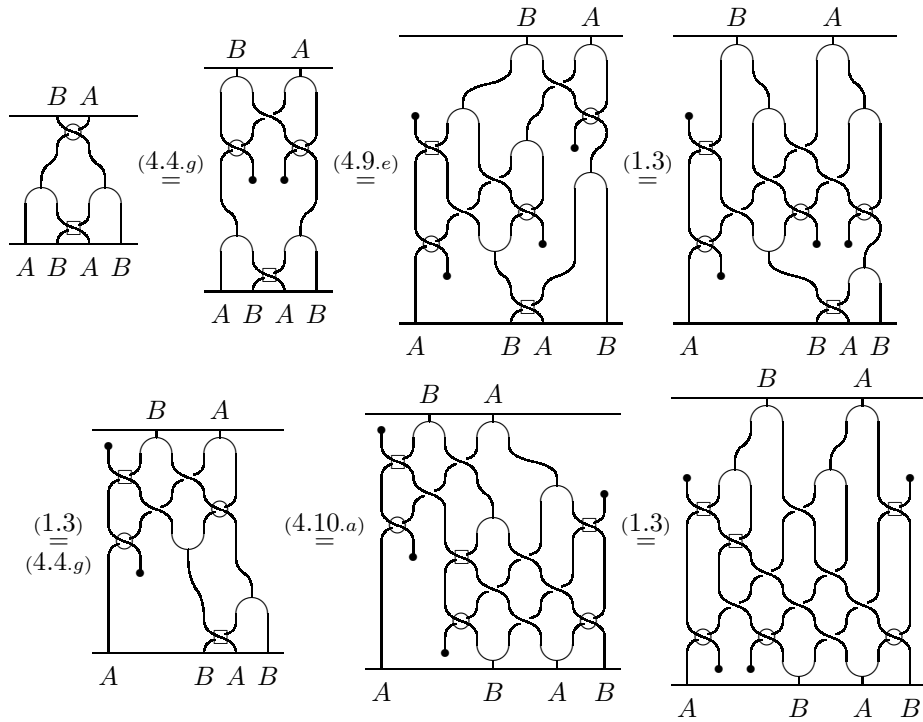
*Proof.* We have already seen in Corollary 4.4 that (i), (ii) and (iii) are equivalent. The equivalences  $(ii) \Leftrightarrow (iv)$  and  $(iii) \Leftrightarrow (v)$  follow from Lemma 4.5.  $(ii) \Rightarrow (vi)$ . We have seen in the proof of Corollary 4.2 that (4.4.a) implies (4.9.a,c), and that (4.4.b) implies (4.9.b,d). (4.4.g) follows after we compose (4.4.d) to the left with  $\varepsilon_A \otimes \varepsilon_B \otimes \text{Id}_{A \otimes B}$ . It remains to be shown that (4.10.a) holds. To this end, we compute that

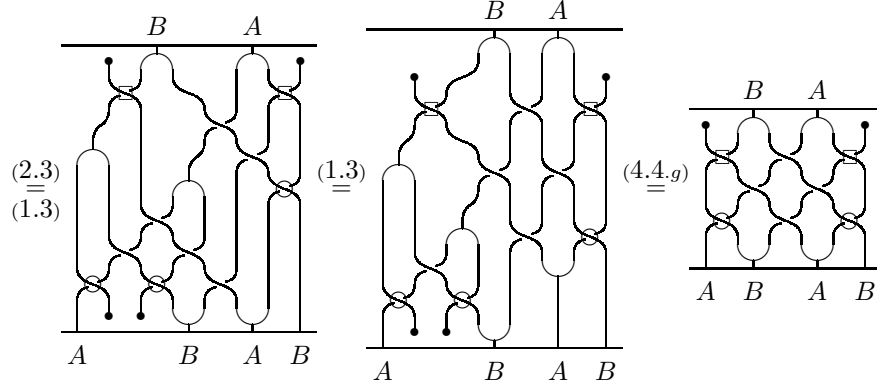
$$\begin{array}{c} A \ B \ A \\ \text{---} \\ \text{[Diagram: A box with two crossings, top-left to bottom-right and top-right to bottom-left]} \\ \text{---} \\ B \ A \ B \end{array} \xrightarrow{(4.4.c)} \begin{array}{c} A \ B \ A \\ \text{---} \\ \text{[Diagram: A box with two crossings, top-left to bottom-right and top-right to bottom-left]} \\ \text{---} \\ B \ A \ B \end{array} \xrightarrow{(4.9.c)} \begin{array}{c} A \ B \ A \\ \text{---} \\ \text{[Diagram: A box with two crossings, top-left to bottom-right and top-right to bottom-left]} \\ \text{---} \\ B \ A \ B \end{array}$$



as needed. In the last but one equality we also applied the naturality of the braiding to the morphism  $\psi$ .

(vi)  $\Rightarrow$  (ii). It is easy to see that (4.10.a) implies (4.9.c,f). We know from Lemma 4.5 that (4.9.a,c) imply (4.4.a), and that (4.9.b,d) imply (4.4.b). (4.4.d) can be proved as follows:





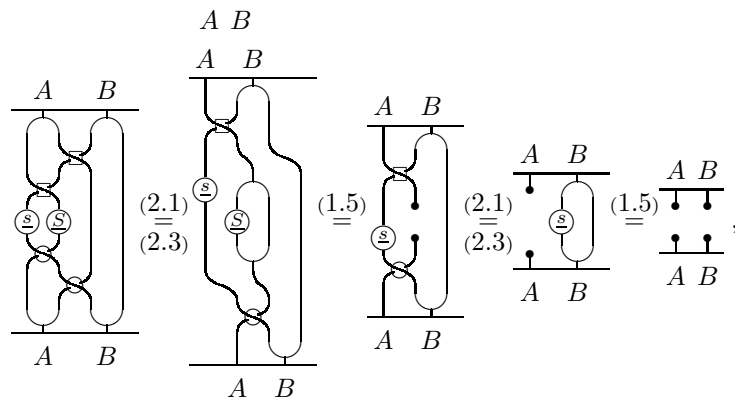
Finally, we observe that the proof of  $(iii) \Leftrightarrow (vii)$  is similar to the proof of  $(ii) \Leftrightarrow (vi)$ .  $\square$

If  $C$  is a coalgebra and  $\mathbb{A}$  is an algebra, then  $\text{Hom}_C(C, \mathbb{A})$  is a monoid, with the convolution  $f * g = \underline{m}_{\mathbb{A}}(f \otimes g)\underline{\Delta}_C$  as multiplication, and unit  $\underline{\eta}_{\mathbb{A}}\underline{\varepsilon}_C$ . We will now discuss some sufficient conditions for a cross product bialgebra to be a Hopf algebra.

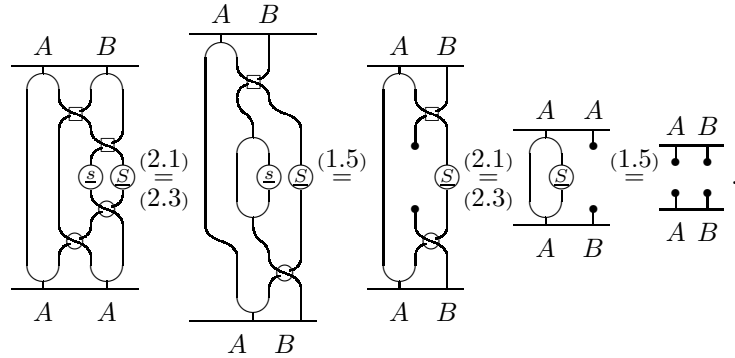
**Proposition 4.7.** *Let  $A \times_{\psi}^{\phi} B$  be a cross product bialgebra, and assume that  $\text{Id}_A$  and  $\text{Id}_B$  are convolution invertible. Then  $A \times_{\psi}^{\phi} B$  is a Hopf algebra.*

*Proof.* Let  $\underline{S}$  be the convolution inverse of  $\text{Id}_A$ , and  $\underline{s}$  the convolution inverse of

$\text{Id}_B$ . We then claim that  $\begin{array}{c} A \ B \\ \hline \underline{s} \ \underline{S} \\ \hline A \ B \end{array}$  is the antipode for  $A \times_{\psi}^{\phi} B$ . Indeed, we have

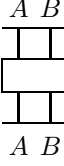


and



$\square$

*Remark 4.8.* Some of the sufficient conditions in Proposition 4.7 are also necessary.

More precisely, if  $A \times_{\psi}^{\phi} B$  admits  as antipode, then (1.5) specializes to

$$(4.11) \quad \begin{array}{c} \text{A} \quad \text{B} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{A} \quad \text{B} \end{array} = \begin{array}{c} \text{A} \quad \text{B} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{A} \quad \text{B} \end{array} = \begin{array}{c} \text{A} \quad \text{B} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{A} \quad \text{B} \end{array}.$$

If we compose the first equality to the left with  $\varepsilon_A \otimes \text{Id}_B$  and to the right with  $\eta_A \otimes \text{Id}_B$ , and the second equality to the left with  $\text{Id}_A \otimes \varepsilon_B$  and to the right with  $\text{Id}_A \otimes \eta_B$ , we deduce that

$$(4.12) \quad \begin{array}{c} \text{B} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{B} \end{array} = \begin{array}{c} \text{B} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{B} \end{array} \quad \text{and} \quad \begin{array}{c} \text{A} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{A} \end{array} = \begin{array}{c} \text{A} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{A} \end{array}.$$

This means that  $\text{Id}_B$  has a left inverse in  $\text{Hom}(B, B)$  and that  $\text{Id}_A$  has a right inverse in  $\text{Hom}(A, A)$ . At this moment, it remains unclear to us whether these one-sided inverses are inverses. We will see in Section 7 that this is true in the case of a smash (co)product Hopf algebra.

## 5. CROSS PRODUCT BIALGEBRAS AND HOPF DATA

If  $(A, B, \psi, \phi)$  is a cross product algebra-coalgebra datum, then  $A$  is a left  $B$ -module and a left  $B$ -comodule, and  $B$  is a right  $A$ -module and a right  $A$ -comodule, see Lemmas 2.2 and 2.4. Now we can ask the following question: suppose that  $A$  and  $B$  are algebras and coalgebras, that  $A$  is a left  $B$ -module and a left  $B$ -comodule, and  $B$  is a right  $A$ -module and a right  $A$ -comodule. Is there a list of necessary and sufficient conditions that these actions and coactions need to satisfy, so that they give rise to bialgebra admissible tuple?

This question was partially answered in [3]. In [3, Def. 2.5], a list of axioms is proposed, we call this list the Bespalov-Drabant list. If these axioms are satisfied, then  $(A, B)$  is a Hopf pair. If  $(A, B, \psi, \phi)$  is a bialgebra admissible tuple, then  $(A, B)$ , with actions and coactions given by (2.2-2.4), is a Hopf pair, see [3, Prop. 2.7]. Moreover, the - crucial - conditions (4.4.g,c) show that  $\psi$  and  $\phi$  can be recovered from the actions and coactions.

Conversely, given a Hopf pair, we can produce a cross product algebra-coalgebra datum  $(A, B, \psi, \phi)$ , but we don't know whether it is a bialgebra admissible tuple, see [3, Prop. 2.6]. Otherwise stated, we obtain a cross product algebra and coalgebra, but we don't know whether it is a bialgebra. We could also say the following: the Bespalov-Drabant list is necessary, but not sufficient. Using the results of Section 4, we are able to present an alternative list of necessary and sufficient conditions. Basically, this is a - technical - restatement of Theorem 4.6. The computations will

turned out to be quite lengthy, and this is why we decided to divide them over several Lemmas.

**Lemma 5.1.** *Let  $A, B$  be algebras and coalgebras such that  $\varepsilon_X \circ \eta_X = \text{Id}_1$ ,  $\varepsilon_X \circ \underline{m}_X = \varepsilon_X \otimes \varepsilon_X$  and  $\underline{\Delta}_X \circ \eta_X = \eta_X \otimes \eta_X$ , for all  $X \in \{A, B\}$ . Furthermore, assume that  $\psi : B \otimes A \rightarrow A \otimes B$  and  $\phi : A \otimes B \rightarrow B \otimes A$  are morphisms in  $\mathcal{C}$  satisfying (2.1.c-d) and (2.3.c-d). Then*

$$\begin{aligned}
 (i) \quad & \begin{array}{c} \text{Diagram 1} \\ \text{B A} \\ \text{A A B} \end{array} = \begin{array}{c} \text{Diagram 2} \\ \text{B A} \\ \text{A A B} \end{array} \quad \text{if and only if (4.4.g) and (4.9.e) hold;} \\
 (ii) \quad & \begin{array}{c} \text{Diagram 3} \\ \text{B A} \\ \text{A B B} \end{array} = \begin{array}{c} \text{Diagram 4} \\ \text{B A} \\ \text{A B B} \end{array} \quad \text{if and only if (4.4.g) and (4.9.f) hold;} \\
 (iii) \quad & \begin{array}{c} \text{Diagram 5} \\ \text{A A B} \\ \text{B A} \end{array} = \begin{array}{c} \text{Diagram 6} \\ \text{A A B} \\ \text{B A} \end{array} \quad \text{if and only if (4.4.c) and (4.9.c) hold;} \\
 (iv) \quad & \begin{array}{c} \text{Diagram 7} \\ \text{A B B} \\ \text{B A} \end{array} = \begin{array}{c} \text{Diagram 8} \\ \text{A B B} \\ \text{B A} \end{array} \quad \text{if and only if (4.4.c) and (4.9.d) hold.}
 \end{aligned}$$

*Proof.* We only prove (i). The proof of (ii), (iii) and (iv) is similar. Actually (iii) and (iv) follow from (i) and (ii) by duality arguments.

The direct implication in (i) follows easily by composing the given equality to the left with  $\varepsilon_A \otimes \text{Id}_{A \otimes B}$ , to obtain (4.4.g), and with  $\text{Id}_{A \otimes A} \otimes \varepsilon_B$ , to obtain (4.9.e).

To prove the converse, we compute

$$\begin{array}{c} \text{Diagram 9} \\ \text{B A} \\ \text{A A B} \end{array} \stackrel{(4.4.g)}{=} \begin{array}{c} \text{Diagram 10} \\ \text{B A} \\ \text{A A B} \end{array} \stackrel{(4.9.e)}{=} \begin{array}{c} \text{Diagram 11} \\ \text{B A} \\ \text{A A B} \end{array}$$

as required. Note that we used the coassociativity of  $\underline{\Delta}_B$  and  $\underline{\Delta}_A$  in the third and the fifth equality.  $\square$

Our next aim is to show that (2.1.a-b) are satisfied if (4.4.g), (4.9.a,b,e,f), and

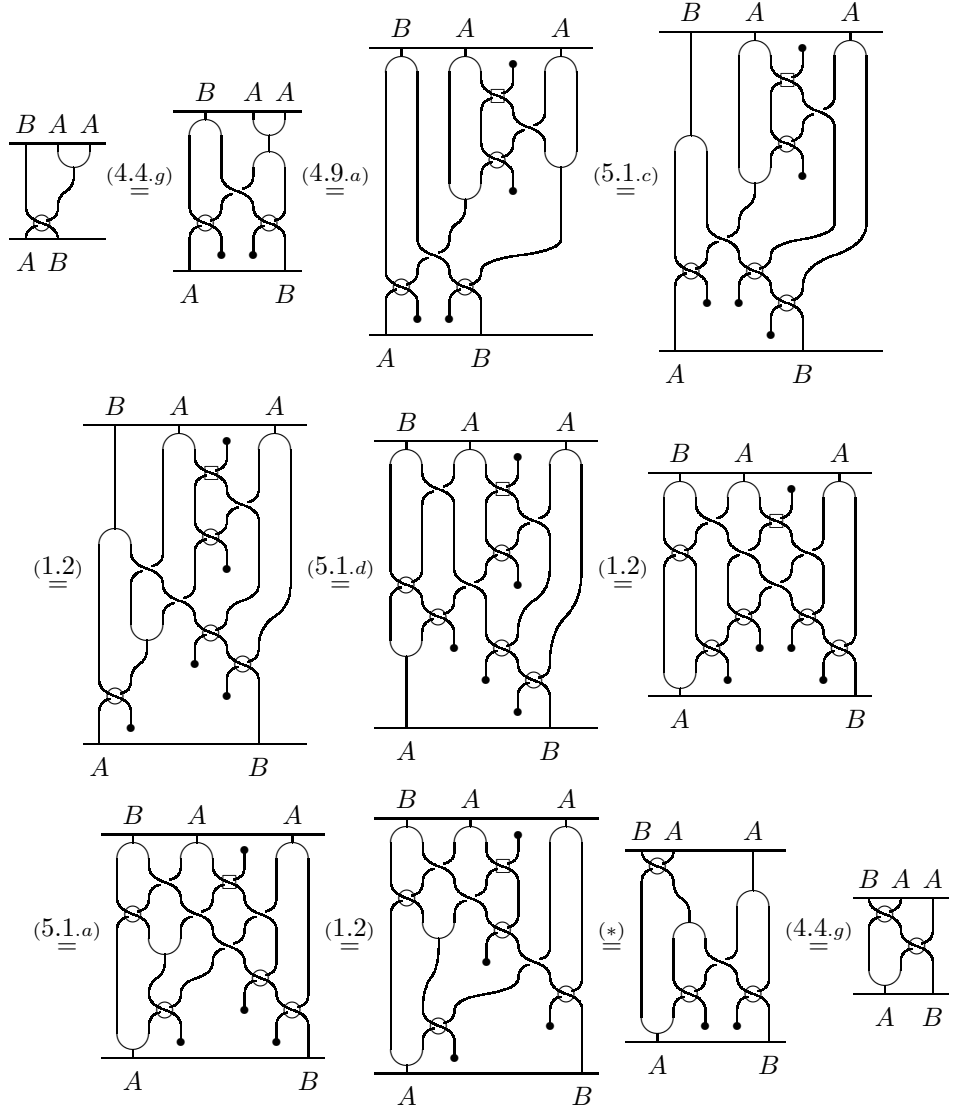
(5.1)

are satisfied. More precisely, we have the following result.

**Lemma 5.2.** *Under the same hypotheses as in Lemma 5.1, we have*

- (i) (4.4.g), (4.9.b,e) and (5.1.a-c) imply (2.1.a);
- (i) (4.4.g), (4.9.a,f) and (5.1.a,c,d) imply (2.1.b).

*Proof.* We prove (ii), the proof of (i) is similar. The proof of (ii) works as follows.



(\*): we used Lemma 5.1 (ii). □

As the reader might expect, we have a dual version of Lemma 5.2. To this end, we need the dual versions of the equations in (5.1), namely,

$$(5.2) \quad \begin{array}{c} \text{(a)} \end{array} \quad \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \ B \ A \end{array} = \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \ B \ A \end{array}, \quad \begin{array}{c} \text{(b)} \end{array} \quad \begin{array}{c} A \\ \hline \text{diagram} \\ \hline B \ B \ A \end{array} = \begin{array}{c} A \\ \hline \text{diagram} \\ \hline B \ B \ A \end{array},$$

$$(c) \quad \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \quad A \quad A \end{array} = \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \quad A \quad A \end{array}, \quad (d) \quad \begin{array}{c} A \\ \hline \text{diagram} \\ \hline B \quad A \quad A \end{array} = \begin{array}{c} A \\ \hline \text{diagram} \\ \hline B \quad A \quad A \end{array}.$$

The proof of Lemma 5.3 is omitted, as it can be obtained from the proof of Lemma 5.2 using duality arguments.

**Lemma 5.3.** *Under the same hypotheses as in Lemma 5.1, we have*

- (i) (4.4 .c), (4.9.b,c), and (5.2.a-c) imply (2.3.a);
- (ii) (4.4 .c), (4.9.a,d), and (5.2.b-d) imply (2.3.b).

**Theorem 5.4.** *Let  $A$  and  $B$  be algebras and coalgebras. There exist  $\psi : B \otimes A \rightarrow A \otimes B$  and  $\phi : A \otimes B \rightarrow B \otimes A$  such that  $(A, B, \psi, \phi)$  is a bialgebra admissible tuple, that is,  $A \#_{\psi}^{\phi} B$  is a cross product bialgebra if and only if the following assertions hold.*

- (i)  $\varepsilon_X \circ \eta_X = \text{Id}_1$ ,  $\varepsilon_X \circ \underline{m}_X = \varepsilon_X \otimes \varepsilon_X$  and  $\Delta_X \circ \eta_X = \eta_X \otimes \eta_X$ , for all  $X \in \{A, B\}$ ;
- (ii)  $A \in {}_B \mathcal{C}$  via  $\frac{B \quad A}{A}$  satisfying  $\frac{B}{A} = \frac{B}{A}$  and  $\frac{B \quad A}{1} = \frac{B \quad A}{1}$ ;
- (iii)  $A \in {}^B \mathcal{C}$  via  $\frac{A}{B \quad A}$  obeying  $\frac{1}{B \quad A} = \frac{1}{B \quad A}$  and  $\frac{A}{B} = \frac{A}{B}$ ;
- (iv)  $B \in {}_{\mathcal{C}} A$  via  $\frac{B \quad A}{B}$  such that  $\frac{A}{B} = \frac{A}{B}$  and  $\frac{B \quad A}{1} = \frac{B \quad A}{1}$ ;
- (v)  $B \in {}^{\mathcal{C}} A$  via  $\frac{B}{B \quad A}$  such that  $\frac{1}{B \quad A} = \frac{1}{B \quad A}$  and  $\frac{B}{A} = \frac{B}{A}$ ;
- (vi) these actions and coactions are compatible in the sense that

$$\begin{array}{c} \begin{array}{c} B \quad A \quad A \\ \hline \text{diagram} \\ \hline A \end{array} = \begin{array}{c} B \quad A \quad A \\ \hline \text{diagram} \\ \hline A \end{array}, \quad \begin{array}{c} A \\ \hline \text{diagram} \\ \hline B \quad A \quad A \end{array} = \begin{array}{c} A \\ \hline \text{diagram} \\ \hline B \quad A \quad A \end{array}, \quad \begin{array}{c} A \quad A \\ \hline \text{diagram} \\ \hline A \quad A \end{array} = \begin{array}{c} A \quad A \\ \hline \text{diagram} \\ \hline A \quad A \end{array}, \\ \\ \begin{array}{c} B \quad B \quad A \\ \hline \text{diagram} \\ \hline B \end{array} = \begin{array}{c} B \quad B \quad A \\ \hline \text{diagram} \\ \hline B \end{array}, \quad \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \quad B \quad A \end{array} = \begin{array}{c} B \\ \hline \text{diagram} \\ \hline B \quad A \quad A \end{array}, \quad \begin{array}{c} B \quad B \\ \hline \text{diagram} \\ \hline B \quad B \end{array} = \begin{array}{c} B \quad B \\ \hline \text{diagram} \\ \hline B \quad B \end{array}; \end{array}$$



(vii) one of the four following sets of three equations is satisfied: (vii.1) =  $\{(5.3), (5.7)\}$ , (vii.2) =  $\{(5.4), (5.8)\}$ , (vii.3) =  $\{(5.5), (5.7.a), (5.8.a)\}$ , (vii.4) =  $\{(5.6), (5.7.a), (5.8.b)\}$ . These four sets are equivalent if (i-vi) are satisfied.

(5.3)

(5.4)

(5.5)

(5.6)

$$\begin{aligned}
(5.7) \quad (a) \quad & \begin{array}{c} \text{A A} \\ \text{B A} \end{array} = \begin{array}{c} \text{A A} \\ \text{B A} \end{array} ; \quad (b) \quad \begin{array}{c} \text{B B} \\ \text{B A} \end{array} = \begin{array}{c} \text{B B} \\ \text{B A} \end{array} ; \\
(5.8) \quad (a) \quad & \begin{array}{c} \text{B A} \\ \text{A A} \end{array} = \begin{array}{c} \text{B A} \\ \text{A A} \end{array} ; \quad (b) \quad \begin{array}{c} \text{B A} \\ \text{B B} \end{array} = \begin{array}{c} \text{B A} \\ \text{B B} \end{array} .
\end{aligned}$$

*Proof.* Suppose that there exist  $\psi$  and  $\phi$  such that  $A \#_{\psi}^{\phi} B$  is a cross product bialgebra. Then  $A$  is a left  $B$ -module and  $B$ -comodule, and  $B$  is a right  $A$ -module and  $A$ -comodule, see (2.2,2.4). (ii)-(v) follow from (4.2), (4.3), (2.1c,d), (2.3c,d), and the fact that  $\varepsilon_X \circ \eta_X = \text{Id}_1$ , for all  $X = A, B$ .

Now observe that it follows from (4.4.g,c) that  $\psi$  and  $\phi$  can be recovered from the actions and coactions

$$(5.9) \quad \psi = \begin{array}{c} \text{B A} \\ \text{A B} \end{array} \quad \text{and} \quad \phi = \begin{array}{c} \text{A B} \\ \text{B A} \end{array} .$$

Then the six formulas in (vi) are reformulations of (5.1.d), (5.2.d), (4.9.a), (5.1.b), (5.2.a) and (4.9.b). In a similar fashion, we have that

- (vii.1) is a reformulation of (4.9.c,d) and (4.4.d);
- (vii.2) is the reformulation of (4.9.e,f) and (4.4.h) in terms of actions and coactions
- (vii.3) follows from (4.9.d), (4.9.e) and (4.10.a);
- (vii.4) is a reformulation of (4.9.c), (4.9.f) and (4.10.b).

It follows from Theorem 4.6 that these sets of conditions are equivalent.

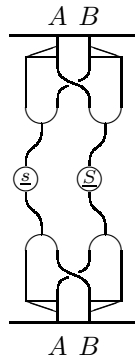
Conversely, assume that  $A$  is a left  $B$ -module and  $B$ -comodule, and that  $B$  is a right  $A$ -module and  $A$ -comodule, satisfying all the conditions of the Theorem. Then we define  $\psi$  and  $\phi$  using (5.9). The actions and coactions are then given by (2.2,2.4) because of the unit-counit conditions in (ii-v). A simple verification tells us that  $\psi$  and  $\phi$  satisfy (4.4.g,c). As in the proof of the direct implication, we show that (i-vi) imply that  $\varepsilon_X \circ \eta_X = \text{Id}_1$  for  $X = A, B$ , and (4.2), (4.3), (2.1c,d), (2.3c,d), (5.1) and (5.2). In addition, the last equalities in (vii.1-vii.4) turn out to be (4.4.d), (4.4.h), (4.10.a) and (4.10.b). (4.9) follows immediately from (vi), using (5.9). It then follows from Lemmas 5.2 and 5.3 that  $(A, B, \psi, \phi)$  is a cross product algebra-coalgebra datum. The result now follows from the equivalence of the conditions (iv-vii) in Theorem 4.6, verification of the details is left to the reader.  $\square$

Let us compare the conditions in Theorem 5.4 with the Bespalov-Drabant list. Conditions (i-v) appear in the Bespalov-Drabant list. Conditions (vi) are also

in the Beshpalov-Drabant list, namely they are the module-algebra, the comodule-coalgebra, and the algebra-coalgebra compatibility. The remaining conditions in the Beshpalov-Drabant list are the module-comodule, module-coalgebra and comodule-algebra compatibility. In order to obtain sufficient conditions, these three conditions have to be replaced by our condition (vii), which appears in four equivalent sets of three equations. Each of the four equations (5.3-5.4) can be regarded as the appropriate substitute of the module-comodule compatibility.

We end this Section with a reformulation of Proposition 4.7 in terms of actions and coactions. The proof is left to the reader.

**Proposition 5.5.** *Let  $A \times_{\psi}^{\phi} B$  be a cross product bialgebra. If  $\text{Id}_A$  and  $\text{Id}_B$  have convolution inverses  $\underline{S}$  and  $\underline{s}$ , then  $A \times_{\psi}^{\phi} B$  is a Hopf algebra with antipode*



## 6. SMASH CROSS (CO)PRODUCT BIALGEBRAS

As a general conclusion so far, we can conclude that there are essentially three ways to describe cross product bialgebras:

- (1) by bialgebra admissible tuples, these are characterized in Theorem 4.6;
- (2) by actions and coactions, this is discussed in Theorem 5.4;
- (3) by injections and projections, this result will be recalled in Proposition 7.1.

The second and third description are not entirely satisfactory in the following sense. As we have remarked above, the substitute of the module-comodule compatibility in Theorem 5.4 appears in four different forms, which are equivalent if some other conditions are satisfied. What is missing is a kind of unified module-comodule compatibility. The objection to the injection/projection description is that we need two algebras/coalgebras and two projections. In some classical results, see a brief survey in the introduction, one projection is sufficient.

In this Section, we will characterize smash product bialgebras and smash coproduct bialgebras, and we will see that the four module-comodule compatibility relations unify in this case.

As applications, we will see that if a cross product bialgebra comes with a tensor product (co)algebra structure then it is necessarily a double cross (co)product bialgebra in the sense of Majid [8]. When we apply this result to the category of sets, then we obtain that the only cross product Hopf algebra structure is the bicross product of groups introduced by Takeuchi in [15]. We will also describe the cross product bialgebras that are a biproduct in the sense of Radford [12].

The second objection can be overcome if we restrict attention to smash (co)product Hopf algebras; then it turns out that one projection suffices, the other one can be recovered from it. This will be the topic of Section 7.

First we will establish that smash product bialgebras and smash coproduct bialgebras are completely determined by normality properties of the morphisms  $\psi$  and  $\phi$ . This is mainly due to the crucial relations (4.9.c,g).

**Definition 6.1.** Let  $A, B$  be algebras and coalgebras and  $\psi : B \otimes A \rightarrow A \otimes B$ ,  $\phi : A \otimes B \rightarrow B \otimes A$  morphisms in  $\mathcal{C}$ .

$$\begin{aligned} \text{(i) } \psi \text{ is called left (right) conormal if } & \frac{\overline{B \ A}}{B} = \frac{B \ A}{B} \left( \frac{\overline{B \ A}}{A} = \frac{B \ A}{A} \right). \\ \text{(ii) } \phi \text{ is called left (right) normal if } & \frac{B}{B \ A} = \frac{B}{B \ A} \left( \frac{A}{B \ A} = \frac{A}{B \ A} \right). \end{aligned}$$

**Lemma 6.2.** Let  $A \times_{\psi}^{\phi} B$  be a cross product bialgebra.  $\psi$  is left (right) conormal if and only if  $A \#_{\psi} B$  is a left (right) smash product algebra.  $\phi$  is left (right) normal if and only if  $A \#^{\phi} B$  is a left (right) smash coproduct coalgebra.

*Proof.* Since  $A \times_{\psi}^{\phi} B$  is a cross product bialgebra the equalities (4.4.g,c) and (4.9.b) hold. Thus if  $\psi$  is left conormal then  $B$  is a bialgebra in  $\mathcal{C}$  and  $\psi$  satisfies (3.1). It then follows from Proposition 3.1 that  $A \times_{\psi} B$  is a smash product. Conversely, if  $A \times_{\psi} B$  is a left smash product algebra, then  $B$  is a bialgebra in  $\mathcal{C}$  and  $\psi$  satisfies (3.1), see Proposition 3.1. Compose (3.1) to the left with  $\underline{\varepsilon}_A \otimes \text{Id}_B$ ; using (4.3.c), it follows that  $\psi$  is left conormal. The proof of the right handed version is similar, and the second assertion is the dual of the first one.  $\square$

**Corollary 6.3.** A cross product bialgebra  $A \#_{\psi}^{\phi} B$  is a left (right) Radford biproduct (this means that  $A \#_{\psi} B$  is a left (right) smash product algebra and  $A \#^{\phi} B$  is a left (right) smash coproduct coalgebra) if and only if  $\psi$  is left (right) conormal and  $\phi$  is left (right) normal. If, moreover,  $B$  is a Hopf algebra and  $\text{Id}_A$  is convolution invertible, then  $A \#_{\psi}^{\phi} B$  is a Hopf algebra.

Our next aim is to describe smash cross product bialgebras, these are cross product bialgebras with a smash product algebra as underlying algebra. Obviously Radford biproducts are special cases, and this is why we did not provide an explicit construction of the Radford biproduct. Theorem 6.4 is a generalization of [6, Theorem 4.5], where the special case where  $A$  and  $B$  are bialgebras is discussed.

**Theorem 6.4.** Let  $A, B$  be algebras and coalgebras, and  $\psi : B \otimes A \rightarrow A \otimes B$  and  $\phi : A \otimes B \rightarrow B \otimes A$  morphisms in  $\mathcal{C}$  such that  $\psi$  is left normal. The following assertions are equivalent:

- (i)  $A \#_{\psi}^{\phi} B$  is a cross product bialgebra (and therefore a smash cross product bialgebra, by Lemma 6.2).
- (ii)  $B$  is a bialgebra in  $\mathcal{C}$ ,  $A$  is a left  $B$ -module algebra and a left  $B$ -comodule

algebra,  $B$  is a right  $A$ -module and comodule and the following relations hold:

If  $\text{Id}_A$  has a convolution inverse  $\underline{S}$  and  $B$  is a Hopf algebra with antipode  $\underline{s}$ , then  $A \times_{\psi}^{\phi} B$  is a Hopf algebra in  $\mathcal{C}$  with antipode

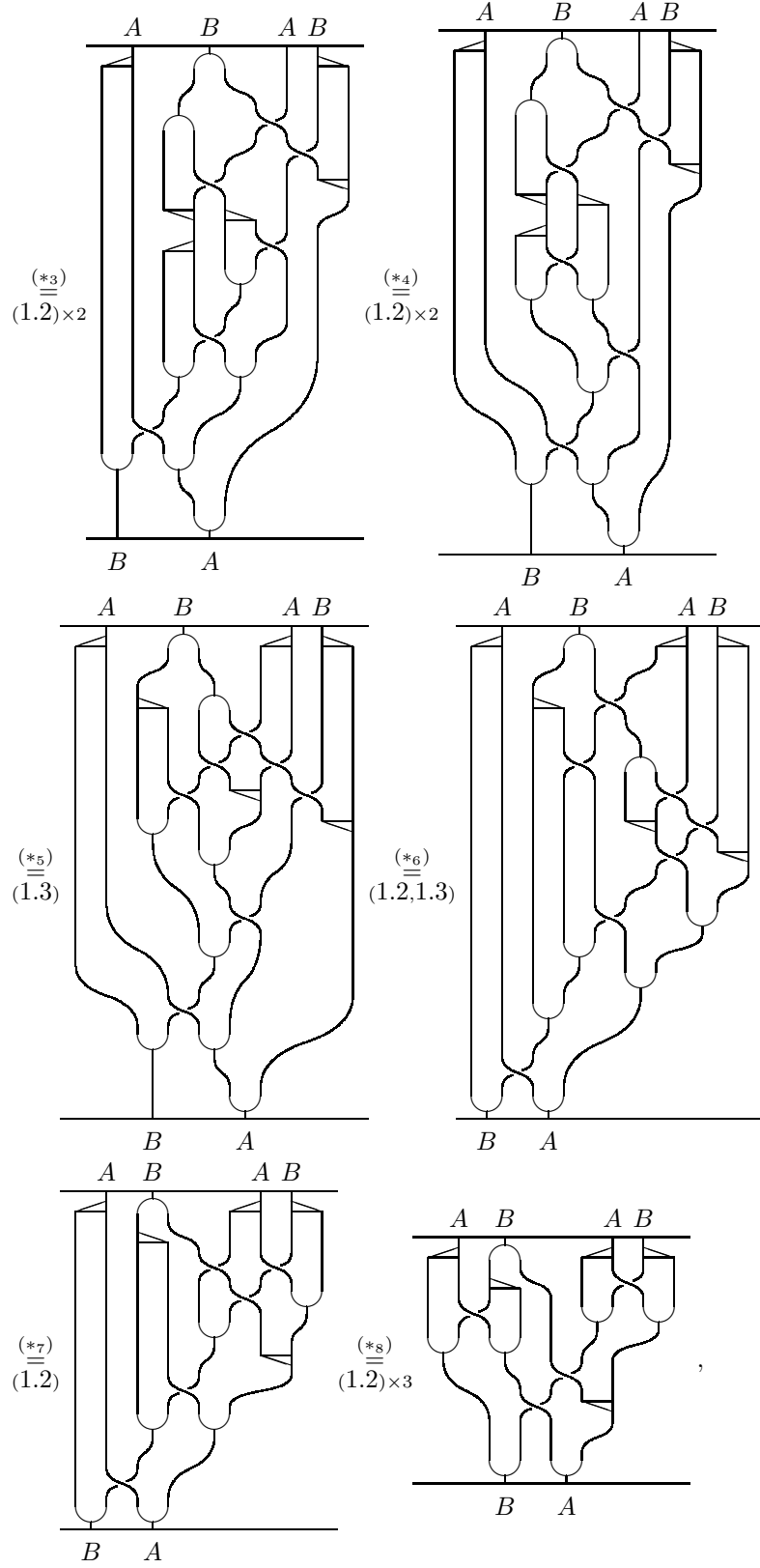
*Proof.*  $A \#_{\psi}^{\phi} B$  is a cross product bialgebra if and only if conditions (i-vi) and (vii.2) from Theorem 5.4 are fulfilled. Using the left normality of  $\psi$ , it follows easily that these conditions reduce to condition (ii) in Theorem 6.4, with one exception: we will show that the third equality in (vii.2) is equivalent to the seventh and eighth compatibility condition in Theorem 6.4 and the fact that  $A$  is a left  $B$ -comodule algebra. Indeed, using the left normality of  $\psi$ , the third equality in (vii.2) takes the

form

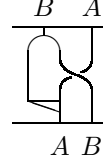
Composing this equality to the right with  $\underline{\eta}_A \otimes \text{Id}_B \otimes \underline{\eta}_A \otimes \text{Id}_B$ , we obtain the seventh compatibility condition. Composing it to the right with  $\text{Id}_A \otimes \underline{\eta}_B \otimes \text{Id}_A \otimes \underline{\eta}_B$ , we find

that the left  $B$ -coaction on  $A$  is a morphism in  ${}^B\mathcal{C}$ . Together with  $\frac{1}{\text{Id}_B \otimes \text{Id}_A} = \frac{1}{\text{Id}_B \otimes \text{Id}_A}$ ,

this tells us that  $A$  is a left  $B$ -comodule algebra. Finally, composition to the right with  $\underline{\eta}_A \otimes \text{Id}_{B \otimes A} \otimes \underline{\eta}_B$  gives the eighth compatibility condition. The proof on the converse implication is based on a direct computation:



We used the following properties: At  $(*_1)$ :  $A$  is a left  $B$ -comodule algebra, and the seventh compatibility condition; at  $(*_2)$ :  $\underline{\Delta}_B$  is coassociative and  $\underline{m}_A$  is associative; at  $(*_3)$  and  $(*_8)$ :  $\underline{m}_A$  and  $\underline{m}_B$  are associative; at  $(*_4)$  and  $(*_6)$ :  $\underline{m}_B$  is associative; at  $(*_5)$ :  $\underline{\Delta}_B$  coassociative, and the eighth compatibility condition; at  $(*_7)$ : naturality

of the braiding,  $A$  is a left  $B$ -comodule algebra, and the fact that  is a

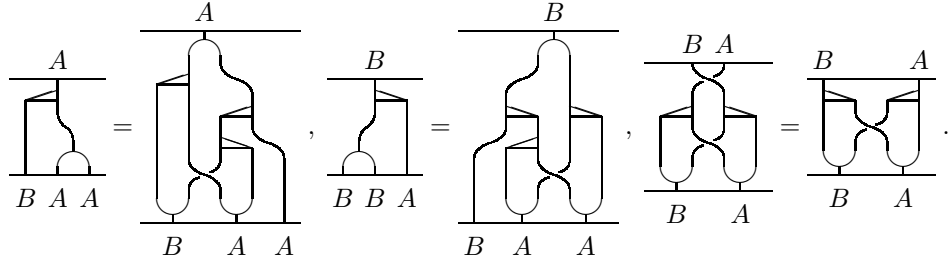
morphism in  $\mathcal{C}$ . The assertion concerning the antipode of  $A \times_{\psi}^{\phi} B$  follows easily from Proposition 4.7.  $\square$

Obviously we also have a right handed version of Theorem 6.4.

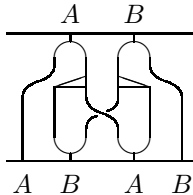
**Corollary 6.5.** *If  $A \times_{\psi}^{\phi} B$  is a cross product bialgebra, and  $\psi$  is left and right conormal then  $A \times_{\psi}^{\phi} B = A \blacktriangleright B$  is a double cross coproduct bialgebra. If  $A$  and  $B$  are Hopf algebras, then  $A \blacktriangleright B$  is also a Hopf algebra.*

*Proof.* First observe that  $\psi$  is left and right conormal if and only if  $\psi$  is equal to the braiding of  $B$  and  $A$  in  $\mathcal{C}$ : one implication follows from (4.4.g), and the other one is immediate. Then (ii) in Theorem 6.4 takes the form:

- 1)  $A, B$  are bialgebras such that  $A$  is a left  $B$ -comodule algebra and  $B$  is a right  $A$ -comodule algebra;
- 2) The following equalities hold:



In this situation, the algebra structure of  $A \times_{\psi}^{\phi} B$  is the tensor product algebra of  $A$  and  $B$ , while the coalgebra structure is given by



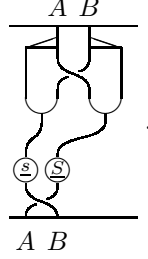
$$\underline{\Delta}_{A \times_{\psi}^{\phi} B} =$$

and  $\underline{\varepsilon}_{A \times_{\psi}^{\phi} B} = \underline{\varepsilon}_A \otimes \underline{\varepsilon}_B$ . This tells us that  $A \times_{\psi}^{\phi} B = A \blacktriangleright B$  is a double cross coproduct bialgebra.

Finally, if  $A$  and  $B$  are Hopf algebras with antipodes  $\underline{S}$  and  $\underline{s}$ , then  $A \blacktriangleright B$  is a



Hopf algebra with antipode



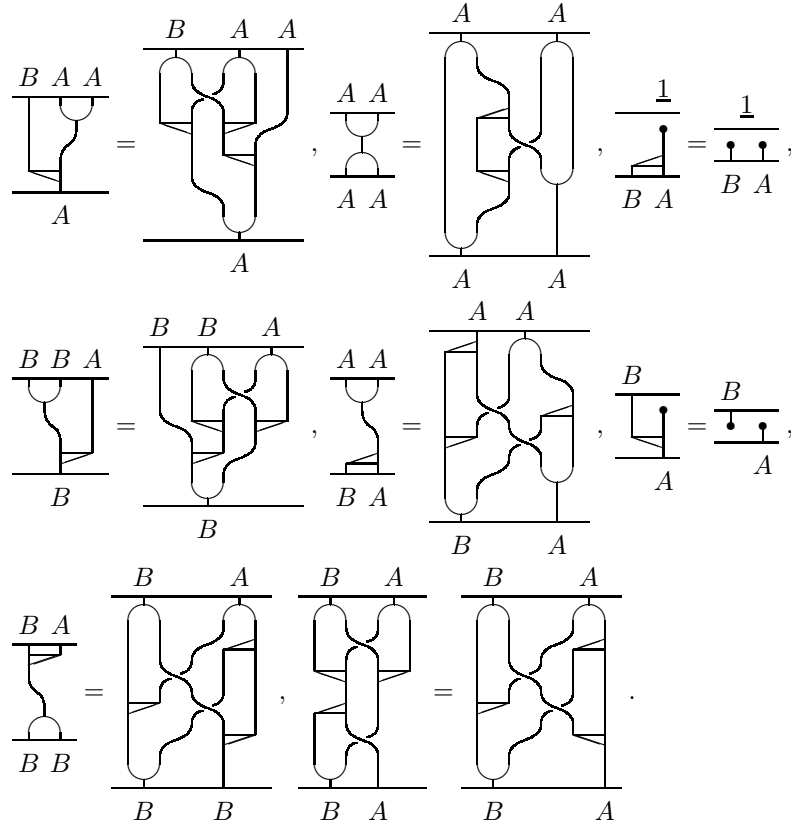
□

Now we investigate the dual situation. A cross product bialgebra  $A \#_\psi^\phi B$  is called a smash cross coproduct bialgebra if  $A \#^\phi B$  is a smash product coalgebra.

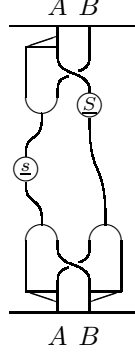
**Theorem 6.6.** *Let  $A, B$  be algebras and coalgebras, and  $\psi : B \otimes A \rightarrow A \otimes B$  and  $\phi : A \otimes B \rightarrow B \otimes A$  morphisms in  $\mathcal{C}$  such that  $\phi$  is left normal. Then the following assertions are equivalent.*

(i)  $A \#_\psi^\phi B$  is a cross product bialgebra (and therefore a smash cross coproduct bialgebra, by Lemma 6.2).

(ii)  $B$  is a bialgebra,  $A$  is a left  $B$ -comodule coalgebra, a left  $B$ -module coalgebra, a right  $B$ -module and a right  $B$ -comodule and the following compatibility relations hold:



If  $B$  is a Hopf algebra with antipode  $\underline{s}$  and  $\text{Id}_A$  has a convolution inverse  $\underline{S}$ , then  $A \#_\psi^\phi B$  is a Hopf algebra with antipode

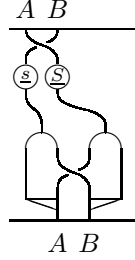


makes  $A \times_\psi^\phi B$  a Hopf algebra in  $\mathcal{C}$ .

*Proof.* We omit the proof, as it is merely a dual version of the proof of Theorem 6.4. Let us just mention that the left normality of  $\phi$  implies that the conditions (i-vi) and (vii.1) in Theorem 5.4 are equivalent to the eight compatibility conditions in the present Theorem.  $\square$

We invite the reader to state the right handed version of Theorem 6.6. Combining the left and right handed versions of Theorem 6.6, we can characterize cross product bialgebras having the property that  $\phi$  is left and right normal.

**Corollary 6.7.** *Let  $A \#_\psi^\phi B$  be a cross product bialgebra such that  $\phi$  is left and right normal. Then  $(A, B)$  is a right-left matched pair and  $A \#_\psi^\phi B = A \bowtie B$ , the double cross product bialgebra associated to  $(A, B)$ . If  $A$  and  $B$  are Hopf algebras, then  $A \bowtie B$  is also a Hopf algebra, with antipode*



*Proof.* It can be easily seen from (4.4.c) that  $\phi$  is left and right normal if and only if it is equal to the braiding of  $A$  and  $B$ . The rest of the proof is then similar to the proof of Corollary 6.5. We obtain relations that tell us that  $(A, B)$  is a right-left matched pair. Moreover,  $A \#^\phi B$  is the tensor product coalgebra, and  $A \#_\psi^\phi B$  is a double cross product bialgebra.  $\square$

We refer to [7, 15] for detail on the bicross product of two groups.

**Corollary 6.8.** *A cross product Hopf algebra in the category of sets is a bicross product of two groups.*

*Proof.* It is well-known that an algebra in Sets is a monoid, and that any set  $X$  has a unique coalgebra structure given by the comultiplication  $\underline{\Delta}_X(x) = (x, x)$ , for all  $x \in X$ , and the counit  $\underline{\varepsilon}_X = *$ , where the singleton  $\{*\}$  is the unit object of the monoidal category Sets. In this way any monoid  $M$  is a bialgebra in Sets and it is, moreover, a Hopf algebra if and only if  $M$  is a group. Consequently, the only cross

coproduct in Sets is the tensor product coalgebra, and the statement then follows from Corollary 6.7.  $\square$

## 7. THE STRUCTURE OF A HOPF ALGEBRA WITH AN APPROPRIATE PROJECTION

As we have already mentioned several times, cross product bialgebras can be characterized using injections and projections. We now recall this classical result, see [3, Prop. 2.2], [6, Theorem 4.3], with a sketch of proof.

**Proposition 7.1.** *For a bialgebra  $H$ , the following statements are equivalent:*

- (i)  $H$  is isomorphic to a cross product bialgebra;
- (ii) There exist algebras and coalgebras  $A, B$  and morphisms  $i : B \rightarrow H, \pi : H \rightarrow B, j : A \rightarrow H, p : H \rightarrow A$  such that
  - $i, j$  are algebra morphisms,  $p, \pi$  are coalgebra morphisms and  $pj = \text{Id}_A$  and  $\pi i = \text{Id}_B$ ;
  - $\zeta = \underline{m}_H(j \otimes i) : A \otimes B \rightarrow H$  is an isomorphism in  $\mathcal{C}$  with inverse  $\zeta^{-1} = (p \otimes \pi) \underline{\Delta}_H : H \rightarrow A \otimes B$ .

*Proof.* For the complete proof, we refer to [3]. For later reference, we give a brief sketch of the proof of (ii)  $\Rightarrow$  (i).  $\psi$  and  $\phi$  are defined by the formulas

$$(7.1) \quad \psi = \begin{array}{c} \overline{B \ A} \\ \downarrow \\ \begin{array}{c} \textcircled{i} \quad \textcircled{j} \\ \downarrow \\ \textcircled{p} \quad \textcircled{\pi} \\ \downarrow \\ \overline{A \ B} \end{array} \end{array} \quad \text{and} \quad \phi = \begin{array}{c} \overline{A \ B} \\ \downarrow \\ \begin{array}{c} \textcircled{j} \quad \textcircled{i} \\ \downarrow \\ \textcircled{\pi} \quad \textcircled{p} \\ \downarrow \\ \overline{B \ A} \end{array} \end{array}.$$

Then we show that  $A \#_{\psi}^{\phi} B$  is a cross product bialgebra, and that  $\zeta$  is an isomorphism of bialgebras.  $\square$

In Proposition 7.1, we need two data, namely  $(A, p, j)$  and  $(B, \pi, i)$ . We will see that one of the two data can be recovered from the other one if some additional conditions are satisfied.

**Lemma 7.2.** *Let  $H = A \#_{\psi}^{\phi} B$  be a (left) smash cross product bialgebra and  $\pi = \frac{H}{B}$  and  $i = \frac{B}{H}$  the canonical morphisms. Then the following assertions hold.*

- (i)  $\pi$  is a bialgebra morphism,  $i$  is an algebra morphism,  $\pi i = \text{Id}_B$  and

$$(7.2) \quad (a) \quad \begin{array}{c} \overline{B} \\ \downarrow \\ \begin{array}{c} \textcircled{i} \\ \downarrow \\ \textcircled{\pi} \end{array} \\ \downarrow \\ \overline{H \ B} \end{array} = \begin{array}{c} \overline{B} \\ \downarrow \\ \begin{array}{c} \textcircled{i} \\ \downarrow \\ \textcircled{\pi} \end{array} \\ \downarrow \\ \overline{H \ B} \end{array} \quad \text{and} \quad (b) \quad \begin{array}{c} \overline{B} \\ \downarrow \\ \begin{array}{c} \textcircled{i} \\ \downarrow \\ \textcircled{s} \\ \downarrow \\ \textcircled{i} \end{array} \\ \downarrow \\ \overline{H \ H} \end{array} = \begin{array}{c} \overline{B} \\ \downarrow \\ \begin{array}{c} \textcircled{i} \\ \downarrow \\ \textcircled{\pi} \\ \downarrow \\ \textcircled{i} \end{array} \\ \downarrow \\ \overline{H \ H} \end{array}.$$

For (7.2.b), we need the additional assumption that  $B$  is a Hopf algebra, with antipode  $\underline{s}$ .

(ii) If  $H$  is a Hopf algebra with antipode  $\underline{S} = \begin{array}{c} A \ B \\ \hline \boxed{\phantom{0}} \\ \hline A \ B \end{array}$  then  $B$  is a Hopf algebra with antipode  $\underline{s}$  defined in (7.3.a), satisfying (7.3.b)

$$(7.3) \quad (a) \quad \underline{s} = \begin{array}{c} B \\ \hline \boxed{\phantom{0}} \\ \hline B \end{array}; \quad (b) \quad \begin{array}{c} B \\ \hline i \\ \hline \pi \circ \underline{S} \\ \hline i \\ \hline H \end{array} = \begin{array}{c} B \\ \hline \boxed{\phantom{0}} \\ \hline B \end{array}.$$

*Proof.* The proof of (i) is straightforward, and is left to the reader. Observe that the conormality of  $\psi$  is needed in order to show that  $\pi$  is a bialgebra morphism, but is not needed in the proof of (7.2). (7.2.a) tells us that  $i : B \rightarrow H$  is right  $B$ -colinear. Here  $H \in \mathcal{C}^B$  via  $\pi \circ \underline{\Delta}_H$  and  $B \in \mathcal{C}^B$  via  $\underline{\Delta}_B$ .

We will only prove that the morphism  $\underline{s}$  as defined in (7.3) is antipode for  $B$ . We have seen in Remark 4.8 that  $\text{Id}_B$  has always a left convolution inverse. We prove that it also has a right inverse. Compose (4.11.a) to the left with  $\underline{\varepsilon}_A \otimes \text{Id}_B$  and to

the right with  $\text{Id}_A \otimes \underline{\eta}_B$ . Using the left conormality of  $\psi$  we obtain that  $\begin{array}{c} A \\ \hline \boxed{\phantom{0}} \\ \hline B \end{array} = \begin{array}{c} A \\ \hline \bullet \\ \hline B \end{array}$ .

Now compose (4.11.b) to the left with  $\underline{\varepsilon}_A \otimes \text{Id}_B$  and to the right with  $\underline{\eta}_A \otimes \text{Id}_B$ . Again using the left conormality of  $\psi$ , we now find that

$$\begin{array}{c} B \\ \hline \bullet \\ \hline B \end{array} = \begin{array}{c} B \\ \hline \boxed{\phantom{0}} \\ \hline B \end{array} = \begin{array}{c} B \\ \hline \text{Diagram 1} \\ \hline B \end{array} \stackrel{(1.6)}{=} \begin{array}{c} B \\ \hline \text{Diagram 2} \\ \hline B \end{array} = \begin{array}{c} B \\ \hline \boxed{\phantom{0}} \\ \hline B \end{array}.$$

This shows that  $\underline{s}$ , as defined in (7.3.a), is a right inverse for  $\text{Id}_B$  in  $\text{Hom}(B, B)$ .  $\square$

In Theorem 7.6 we will show that Lemma 7.2 has a converse, at least if some additional technical assumptions are satisfied. In the sequel, we assume that  $B$  is a Hopf algebra,  $H$  is a bialgebra and  $B \xrightleftharpoons[\pi]{i} H$  are morphisms in  $\mathcal{C}$  such that  $\pi$  is a bialgebra morphism,  $i$  is an algebra morphism,  $\pi i = \text{Id}_B$  and (7.2.a,b) hold. At some places, we will consider the situation where  $H$  is also a Hopf algebra, and then we will assume that (7.3.b) holds as well. In addition, we assume that

$(\text{Id}_H \otimes \pi)\underline{\Delta}_H, \text{Id}_H \otimes \underline{\eta}_B : H \rightarrow H \otimes B$  have a equalizer in  $\mathcal{C}$ . This means that there exists  $A \in \mathcal{C}$  and  $j : A \rightarrow H$  such that

$$(7.4) \quad \begin{array}{c} A \\ \hline \textcircled{j} \\ \downarrow \\ \textcircled{\pi} \\ \hline H \quad B \end{array} = \frac{A}{\textcircled{j} \quad \textcircled{\pi}} \cdot \frac{A}{H \quad B}.$$

$(A, j)$  is universal in the following sense: if  $f : X \rightarrow H$  is such that  $(\text{Id}_H \otimes \pi)\underline{\Delta}_H f = (\text{Id}_H \otimes \underline{\eta}_B)f$ , there is a unique morphism  $\tilde{f} : X \rightarrow A$  such that  $j\tilde{f} = f$ . Under these assumptions, we can show that  $H$  is (isomorphic to) a smash cross product bialgebra. The proof of Theorem 7.6 consists of several steps, and we have divided them over the subsequent Lemmas.

**Lemma 7.3.** *Let  $B, H, \pi, i$  be as above. Then  $A$  has an algebra structure such that  $j : A \rightarrow H$  is an algebra morphism.*

*Proof.* Applying the universal property of the equalizer  $(A, j)$ , we find unique morphisms  $\underline{m}_A : A \otimes A \rightarrow A$  and  $\underline{\eta} : \underline{1} \rightarrow A$  morphisms in  $\mathcal{C}$  making the diagrams

$$(7.5) \quad \begin{array}{ccc} A & \xrightarrow{j} & H \\ \swarrow \underline{m}_A & & \uparrow \underline{m}_H(j \otimes j) \\ & A \otimes A & \end{array} \xrightarrow[\text{Id}_H \otimes \underline{\eta}_B]{(\text{Id}_H \otimes \pi)\underline{\Delta}_H} H \otimes B \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{j} & H \\ \swarrow \underline{\eta}_A & & \uparrow \underline{\eta}_H \\ & \underline{1} & \end{array} \xrightarrow[\text{Id}_H \otimes \underline{\eta}_B]{(\text{Id}_H \otimes \pi)\underline{\Delta}_H} H \otimes B.$$

commutative, which means that  $j\underline{m}_A = \underline{m}_H(j \otimes j)$  and  $j\underline{\eta}_A = \underline{\eta}_H$ . Furthermore, a simple inspection shows that  $j\underline{m}_A(\underline{m}_A \otimes \text{Id}_A) = j\underline{m}_A(\text{Id}_A \otimes \underline{m}_A)$  and  $j\underline{m}_A(\text{Id}_A \otimes \underline{\eta}_A) = j = j(\underline{\eta}_A \otimes \text{Id}_A)$ . Since  $j$  is a monomorphism in  $\mathcal{C}$  we deduce that  $\underline{m}_A$  is associative and  $\underline{\eta}_A$  has the unit property. This shows that  $A$  is an algebra and  $j$  is an algebra morphism.  $\square$

The next step is more complicated, and consists in proving that  $A$  also has a coalgebra structure. We will need an extra assumption, namely that  $A \in \mathcal{C}$  is flat:  $- \otimes A$  and  $A \otimes -$  preserve equalizers. Actually, we need that  $\text{Id}_A \otimes j$  and  $j \otimes \text{Id}_A$  are monomorphisms, in order to obtain that  $j \otimes j$  is a monomorphism.

**Lemma 7.4.** *Let  $A, B, H, \pi, i, j$  be as above, and consider  $\tilde{p} = \underline{m}_H(\text{Id}_H \otimes i\underline{\pi})\underline{\Delta}_H : H \rightarrow H$ . Then there exists a morphism  $p : H \rightarrow A$  such that  $jp = \tilde{p}$  and  $pj = \text{Id}_A$ . Furthermore,  $A$  is a coalgebra, and  $p$  is a coalgebra morphism.*

*Proof.* We compute that

It follows from the universal property of the equalizer  $(A, j)$  that there exists a unique morphism  $p : H \rightarrow A$  such that  $jp = \tilde{p}$ . Then  $jpj = \tilde{p}j$ . From (7.4), we deduce that  $\tilde{p}j = j$ , so  $jpj = j$ , hence  $pj = \text{Id}_A$ .

Now we construct the coalgebra structure on  $A$ . We claim that  $(A, p)$  is the coequalizer of  $\underline{m}_H(\text{Id}_H \otimes i)$ ,  $\text{Id}_H \otimes \underline{\varepsilon}_B : H \otimes B \rightarrow B$ . First of all, we have that

$$jp(\underline{m}_H(\text{Id}_H \otimes i)) = \tilde{p}(\underline{m}_H(\text{Id}_H \otimes i)) = \tilde{p}(\text{Id}_H \otimes \underline{\varepsilon}_B) = jp(\text{Id}_H \otimes \underline{\varepsilon}_B),$$

since

Secondly, we need to prove the universal property. Assume that  $\tilde{f} : H \rightarrow X$  is such that  $\tilde{f}\underline{m}_H(\text{Id}_H \otimes i) = \tilde{f}(\text{Id}_H \otimes \underline{\varepsilon}_B)$ . We have to show that there is a unique  $f : A \rightarrow X$  such that  $fp = \tilde{f}$ . If  $f$  exists, then  $f = f\text{Id}_A = fpj = \tilde{f}j$ , hence  $f$  is unique. To prove the existence, let  $f = \tilde{f}j$ , then

$$fp = \tilde{f}jp = \tilde{f}\tilde{p} = \tilde{f}\underline{m}_H(\text{Id}_H \otimes i)(\text{Id}_H \otimes \underline{\varepsilon}_B)\underline{\Delta}_H = \tilde{f}(\text{Id}_H \otimes \underline{\varepsilon}_B)\underline{\Delta}_H = \tilde{f},$$

Now we use the universal property of the coequalizer to construct the comultiplication on  $A$ . First observe that

where we freely used associativity and coassociativity of the multiplications and comultiplications that are involved, and the fact the  $\pi$  is a bialgebra morphism and

$i$  is an algebra morphism. So we have shown that

$$(j \otimes j)\tilde{f}\underline{m}_H(\text{Id}_H \otimes i) = (\tilde{p} \otimes \tilde{p})\underline{\Delta}_H(\text{Id}_H \otimes \underline{\varepsilon}_B) = (j \otimes j)\tilde{f}(\text{Id}_H \otimes \underline{\varepsilon}_B).$$

Now  $j \otimes j$  is a monomorphism in  $\mathcal{C}$ , see the notes preceding the Lemma, and it follows that  $\tilde{f}\underline{m}_H(\text{Id}_H \otimes i) = \tilde{f}(\text{Id}_H \otimes \underline{\varepsilon}_B)$ . Applying the universal property of the coequalizer  $(A, p)$ , we find a unique morphism  $\underline{\Delta}_A : A \rightarrow A \otimes A$  such that  $\underline{\Delta}_A p = (p \otimes p)\underline{\Delta}_H$ . Arguments dual to those presented in the proof of Lemma 7.3 show that  $\underline{\Delta}_A$  is coassociative.

Finally,  $\tilde{f} = \underline{\varepsilon}_H : H \rightarrow \underline{1}$  satisfies  $\tilde{f}\underline{m}_H(\text{Id}_H \otimes i) = \tilde{f}(\text{Id}_H \otimes \underline{\varepsilon}_B)$ . Applying the universal property again, we find a unique morphism  $\underline{\varepsilon}_A : A \rightarrow \underline{1}$  such that  $\underline{\varepsilon}_A p = \underline{\varepsilon}_H$ . It is immediate that  $\underline{\varepsilon}_A$  is a counit for  $\underline{\Delta}_A$ , and hence  $(A, \underline{\Delta}_A, \underline{\varepsilon}_A)$  is a coalgebra in  $\mathcal{C}$ . The construction of  $\underline{\Delta}_A$  and  $\underline{\varepsilon}_A$  is such that  $p$  is a coalgebra morphism, and this finishes the proof.  $\square$

Applying the formulas that we obtained above, we easily see that  $\underline{\Delta}_A = \underline{\Delta}_A p j = (p \otimes p)\underline{\Delta}_H j$ . Furthermore

$$(7.8) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \end{array} \quad (7.4)$$

The diagrams represent string diagrams for the morphisms involved. Diagram 1 shows a vertical line from  $A$  to  $j$ , then a horizontal line to  $p$ , which splits into two lines going to  $A$  and  $H$ . Diagram 2 shows a vertical line from  $A$  to  $j$ , then a horizontal line to  $p$ , which splits into two lines going to  $A$  and  $H$ . Diagram 3 shows a vertical line from  $A$  to  $j$ , then a horizontal line to  $p$ , which splits into two lines going to  $A$  and  $H$ . Diagram 4 shows a vertical line from  $A$  to  $j$ , then a horizontal line to  $p$ , which splits into two lines going to  $A$  and  $H$ . Diagram 5 shows a vertical line from  $A$  to  $j$ , then a horizontal line to  $p$ , which splits into two lines going to  $A$  and  $H$ . Diagram 6 shows a vertical line from  $A$  to  $j$ , then a horizontal line to  $p$ , which splits into two lines going to  $A$  and  $H$ . Diagram 7 shows a vertical line from  $A$  to  $j$ , then a horizontal line to  $p$ , which splits into two lines going to  $A$  and  $H$ .

These formulas will be used in Lemma 7.5.

**Lemma 7.5.** *Let  $A, B, H, \pi, i, \tilde{p}, j$  be as in Lemma 7.4. Assume that  $H$  is a Hopf algebra with antipode  $\underline{S}$ , and that (7.3.b) is fulfilled. Then  $\text{Id}_A$  is convolution invertible.*

*Proof.*  $\tilde{f} = \underline{m}_H(i\pi \otimes \underline{S})\underline{\Delta}_H : H \rightarrow H$  satisfies the equation

$$(\text{Id}_H \otimes \pi)\underline{\Delta}_H \tilde{f}$$

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \end{array} = (\text{Id}_H \otimes \underline{\eta}_B)\tilde{f}.$$

The diagrams represent string diagrams for the morphisms involved. Diagram 1 shows a vertical line from  $H$  to  $\pi$ , then a horizontal line to  $\underline{S}$ , which splits into two lines going to  $H$  and  $B$ . Diagram 2 shows a vertical line from  $H$  to  $\pi$ , then a horizontal line to  $\underline{S}$ , which splits into two lines going to  $H$  and  $B$ . Diagram 3 shows a vertical line from  $H$  to  $\pi$ , then a horizontal line to  $\underline{S}$ , which splits into two lines going to  $H$  and  $B$ . Diagram 4 shows a vertical line from  $H$  to  $\pi$ , then a horizontal line to  $\underline{S}$ , which splits into two lines going to  $H$  and  $B$ . Diagram 5 shows a vertical line from  $H$  to  $\pi$ , then a horizontal line to  $\underline{S}$ , which splits into two lines going to  $H$  and  $B$ . Diagram 6 shows a vertical line from  $H$  to  $\pi$ , then a horizontal line to  $\underline{S}$ , which splits into two lines going to  $H$  and  $B$ . Diagram 7 shows a vertical line from  $H$  to  $\pi$ , then a horizontal line to  $\underline{S}$ , which splits into two lines going to  $H$  and  $B$ .

Applying the universal property of the equalizer  $(A, j)$ , we obtain a unique morphism  $\tilde{\underline{S}}_A : H \rightarrow A$  such that  $j\tilde{\underline{S}}_A = \tilde{f}$ . We will show that  $\underline{S} = \tilde{\underline{S}}_A j$  is the



**Theorem 7.6.** *Let  $\mathcal{C}$  be a braided monoidal category with equalizers in which every object is flat. For a Hopf algebra  $H$ , the following assertions are equivalent.*

- (i)  $H$  is isomorphic to a smash cross product Hopf algebra;
- (ii) there exist a Hopf algebra  $B$ , an algebra morphism  $i: B \rightarrow H$  and a Hopf algebra morphism  $\pi: H \rightarrow B$  such that  $\pi i = \text{Id}_B$  and the conditions (7.2) and (7.3.b) are fulfilled.

*Proof.*  $(i) \Rightarrow (ii)$  follows from Lemma 7.2.

$(ii) \Rightarrow (i)$ . Let  $(A, j)$  be the equalizer of  $(\text{Id}_H \otimes \pi)\underline{\Delta}_H, \text{Id}_H \otimes \underline{\eta}_B : H \rightarrow H \otimes B$ . It follows from Lemmas 7.3 and 7.4 that  $A$  is an algebra and a coalgebra, and that there is an algebra morphism  $j : A \rightarrow H$  and a coalgebra morphism  $p : H \rightarrow A$  such that  $pj = \text{Id}_A$ . We claim that  $\zeta = \underline{m}_H(j \otimes i) : A \otimes B \rightarrow H$  and  $\zeta^{-1} = (p \otimes \pi)\underline{\Delta}_H : H \rightarrow A \otimes B$  are inverses. Indeed,  $\zeta^{-1}\zeta(j \otimes \text{Id}_B)$  is equal to

and this implies that  $\zeta^{-1}\zeta = \text{Id}_{A \otimes B}$ . We also have that

Now we apply the implication  $(ii) \Rightarrow (i)$  in Proposition 7.1, and obtain that  $\zeta^{-1} : H \rightarrow A \times_{\psi}^{\phi} B$  is a bialgebra isomorphism, with  $\psi$  and  $\phi$  given by (7.1). Now

that  $H$  is isomorphic to a smash cross product bialgebra.

Finally, according to Lemma 7.5  $\text{Id}_A$  is convolution invertible. Together with the fact that  $B$  is a Hopf algebra, this implies that  $A \times_{\psi}^{\phi} B$  is a Hopf algebra, see Proposition 4.7. Then  $\zeta^{-1}$  is a Hopf algebra isomorphism, completing the proof.  $\square$

We leave it to the reader to formulate the right handed version of Theorem 7.6.

**Corollary 7.7.** [12] *Let  $\mathcal{C}$  be a braided monoidal category with equalizers and assume that every object of  $\mathcal{C}$  is flat. Then a Hopf algebra  $H$  is isomorphic to a biproduct Hopf algebra if and only if there exist a Hopf algebra  $B$  and Hopf algebra morphisms  $i : B \rightarrow H, \pi : H \rightarrow B$  such that  $\pi i = \text{Id}_B$ .*

*Proof.* A biproduct Hopf algebra  $A \times_{\psi}^{\phi} B$  is a smash cross product Hopf algebra for which  $\phi$  is left normal. In this case it is easy to see that the canonical morphism  $i : B \rightarrow H$  is a coalgebra morphism, and therefore a Hopf algebra morphism. Then it can be easily checked that (7.2) and (7.3.b) are automatically satisfied. Conversely, if  $i$ ,  $\pi$  and  $B$  are given as in the Theorem, then condition (ii) of Theorem 7.6 is fulfilled since  $i$  is a Hopf algebra morphism. Hence  $H$  is isomorphic to a smash cross product Hopf algebra. It follows from (7.1) that

$$\phi = \begin{array}{c} A \quad B \\ \circlearrowleft \quad \circlearrowright \\ \text{---} \\ \circlearrowright \quad \circlearrowleft \\ \pi \quad \tilde{p} \\ \text{---} \\ B \quad A \end{array}, \text{ hence } \begin{array}{c} B \\ \circlearrowleft \\ \text{---} \\ \circlearrowright \\ B \quad H \end{array} = \begin{array}{c} B \\ \circlearrowleft \\ \text{---} \\ \pi \quad \tilde{p} \\ \text{---} \\ B \quad H \end{array} = \begin{array}{c} B \\ \circlearrowleft \quad \circlearrowright \\ \pi \quad \circlearrowright \\ \text{---} \\ \pi \quad \circlearrowleft \\ \text{---} \\ B \quad H \end{array} = \begin{array}{c} B \\ \circlearrowleft \quad \circlearrowright \\ \pi \quad \circlearrowright \\ \text{---} \\ \pi \quad \circlearrowleft \\ \text{---} \\ B \quad H \end{array} \stackrel{(1.5)}{=} \begin{array}{c} B \\ \circlearrowleft \\ \text{---} \\ \circlearrowright \\ B \quad H \end{array}.$$

We conclude that  $\phi$  is left normal, and therefore  $H$  is isomorphic to a biproduct Hopf algebra.  $\square$

Now we focus attention to double cross coproduct Hopf algebras. Recall that  $X \in \mathcal{C}$  is called right (left) coflat if  $X \otimes -$  (resp.  $- \otimes X$ ) preserves coequalizers.  $X$  is coflat if it is left and right coflat.

**Corollary 7.8.** [8] *Let  $\mathcal{C}$  be a braided monoidal category with equalizers and assume that every object in  $\mathcal{C}$  is flat and right coflat. A Hopf algebra  $H$  is isomorphic to a double cross coproduct Hopf algebra if and only if there exist a Hopf algebra  $B$  and Hopf algebra morphisms  $i : B \rightarrow H$  and  $\pi : H \rightarrow B$  such that  $\pi i = \text{Id}_B$  and (7.9) holds, i.e., the left adjoint action of  $B$  on  $H$  induced by  $i$  is trivial on the image of  $\tilde{p}$ .*

$$(7.9) \quad \begin{array}{c} B \quad H \\ \circlearrowleft \quad \circlearrowright \\ \text{---} \\ \circlearrowright \quad \circlearrowleft \\ \pi \quad \tilde{p} \\ \text{---} \\ B \quad H \end{array} = \begin{array}{c} B \quad H \\ \circlearrowleft \quad \circlearrowright \\ \text{---} \\ \circlearrowright \quad \circlearrowleft \\ \pi \quad \tilde{p} \\ \text{---} \\ H \end{array}.$$

*Proof.* A double cross coproduct Hopf algebra is a biproduct Hopf algebra for which  $\psi$  is right conormal. By Corollary 7.7, it suffices to verify (7.9). This follows directly from the definitions, we leave the details to the reader.

Conversely, assume that  $i$  and  $\pi$  are given. According to Corollary 7.7,  $H$  is isomorphic to a biproduct Hopf algebra  $A \times_{\psi}^{\phi} B$ . The biproduct Hopf algebra is actually

a double cross coproduct Hopf algebra since  $\psi$  is right conormal. Indeed, we have

The right coflatness of  $B$  implies that  $\text{Id}_B \otimes p$  is an epimorphism. Then the right conormality of  $\psi$  follows from the fact that  $j$  is a monomorphism.  $\square$

To make our story complete, we present the dual version of Theorem 7.6. We need some Lemmas first. Most of the proofs are omitted, as they are dual versions of proofs that we presented above.

**Lemma 7.9.** *Let  $H = A \times_{\psi}^{\phi} B$  be a (left) smash cross coproduct bialgebra and let  $\pi : H \rightarrow B$  and  $i : B \rightarrow H$  be the canonical morphisms.*

(i)  $\pi$  is a coalgebra morphism,  $i$  is a bialgebra morphism,  $\pi i = \text{Id}_B$  and

For the second equality, we need the additional assumption that  $B$  is a Hopf algebra with antipode  $\underline{s}$ .

(ii) If  $H$  is a Hopf algebra with antipode  $\underline{s}$  then  $B$  is also a Hopf algebra and

For the converse of Lemma 7.9, we need additional assumptions:  $\mathcal{C}$  has coequalizers, and every object of  $\mathcal{C}$  is coflat, that is, it is left and right coflat.

**Lemma 7.10.** *Let  $H$  be a bialgebra and let  $B$  be a Hopf algebra with antipode  $\underline{s}$  and suppose that we have a bialgebra morphism  $i : B \rightarrow H$  and a coalgebra morphism  $\pi : H \rightarrow B$  such that  $\pi i = \text{Id}_B$ , such that (7.10) holds. Let  $(A, p)$  be the coequalizer*

of  $\underline{m}_H(\text{Id}_H \otimes i)$ ,  $\text{Id}_H \otimes \underline{\varepsilon}_B : H \otimes B \rightarrow B$ . Then we have the following results.  
(i)  $A$  has a coalgebra structure such that  $p : H \rightarrow A$  is a coalgebra morphism in  $\mathcal{C}$ .  
(ii)  $(A, j)$  is the equalizer of  $(\text{Id}_H \otimes \pi) \underline{\Delta}_H$ ,  $\text{Id}_H \otimes \underline{\eta}_H : H \rightarrow H \otimes B$ , where  $\tilde{j} = \underline{m}_H(\text{Id}_H \otimes i \underline{\pi}) \underline{\Delta}_H$  and  $j$  is defined by commutativity of the diagram

$$\begin{array}{ccccc} H \otimes B & \xrightarrow[\text{Id}_H \otimes \underline{\varepsilon}_B]{\underline{m}_H(\text{Id}_H \otimes i)} & H & \xrightarrow{p} & A \\ & & \downarrow \tilde{j} & \nearrow j & \\ & & H & & \end{array}$$

Consequently  $A$  is an algebra and  $j : A \rightarrow H$  is an algebra morphism.  
(iii) If  $H$  is a Hopf algebra with antipode  $\underline{S}$  satisfying (7.11), then  $\text{Id}_A$  is convolution invertible.

*Proof.* (i) follows from Lemma 7.3 by duality arguments. (ii) We just mention that the algebra structure on  $A$  is obtained using (7.5).  
(iii) Applying the universal property of the coequalizer  $(A, p)$ , we find  $\tilde{\underline{S}}_A : A \rightarrow H$  such that the diagram

$$\begin{array}{ccccc} H \otimes B & \xrightarrow[\text{Id}_H \otimes \underline{\varepsilon}_B]{\underline{m}_H(\text{Id}_H \otimes i)} & H & \xrightarrow{p} & A \\ & \searrow \underline{m}_H(\pi i \otimes \underline{S}) \underline{\Delta}_H & \downarrow & \nearrow \tilde{\underline{S}}_A & \\ & & H & & \end{array}$$

commutes. A straightforward computation shows that  $\underline{S} := p \tilde{\underline{S}}_A$  is the convolution inverse of  $\text{Id}_A$ .  $\square$

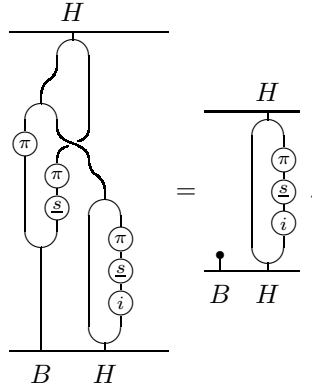
Theorem 7.11 is the dual version of Theorem 7.6. The proof is based on Lemma 7.10 and is omitted.

**Theorem 7.11.** *Let  $\mathcal{C}$  be a braided monoidal category with coequalizers, in which every object of  $\mathcal{C}$  is coflat. A Hopf algebra  $H$  is isomorphic to a smash cross coproduct Hopf algebra if and only if there exists a Hopf algebra  $B$  and morphisms  $i : B \rightarrow H$  and  $\pi : H \rightarrow B$  in  $\mathcal{C}$  such that  $i$  is a Hopf algebra morphism,  $\pi$  is a coalgebra morphism,  $\pi i = \text{Id}_B$  and (7.10-7.11) are satisfied.*

If we specialize Theorem 7.11 to the case where  $\psi$  is left conormal, then we recover Corollary 7.7. This is due to the fact that  $A$  can be defined as an equalizer or as a coequalizer. If we add the condition that  $\phi$  is right conormal, then we obtain Corollary 7.12. The proof is similar to the proof of Corollary 7.8. We just mention that the conditions in (ii) tell us that the left coadjoint coaction of  $B$  on  $H$  induced by  $\pi$  is trivial on the image of  $\tilde{j} = jp$ .

**Corollary 7.12.** *Let  $\mathcal{C}$  be a braided monoidal category with coequalizers, in which every object coflat and left flat. A Hopf algebra  $H$  is isomorphic to a double cross product Hopf algebra if and only if there exist a Hopf algebra  $B$  and Hopf algebra*

morphisms  $i : B \rightarrow H$ ,  $\pi : H \rightarrow B$  such that  $\pi i = \text{Id}_B$  and



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